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Lecture 2
Automata Theory

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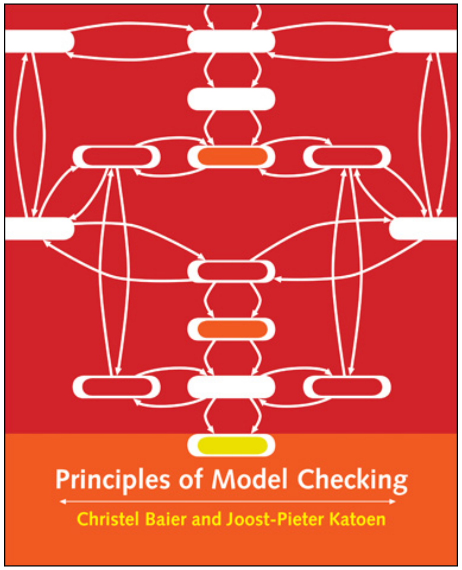
Outline

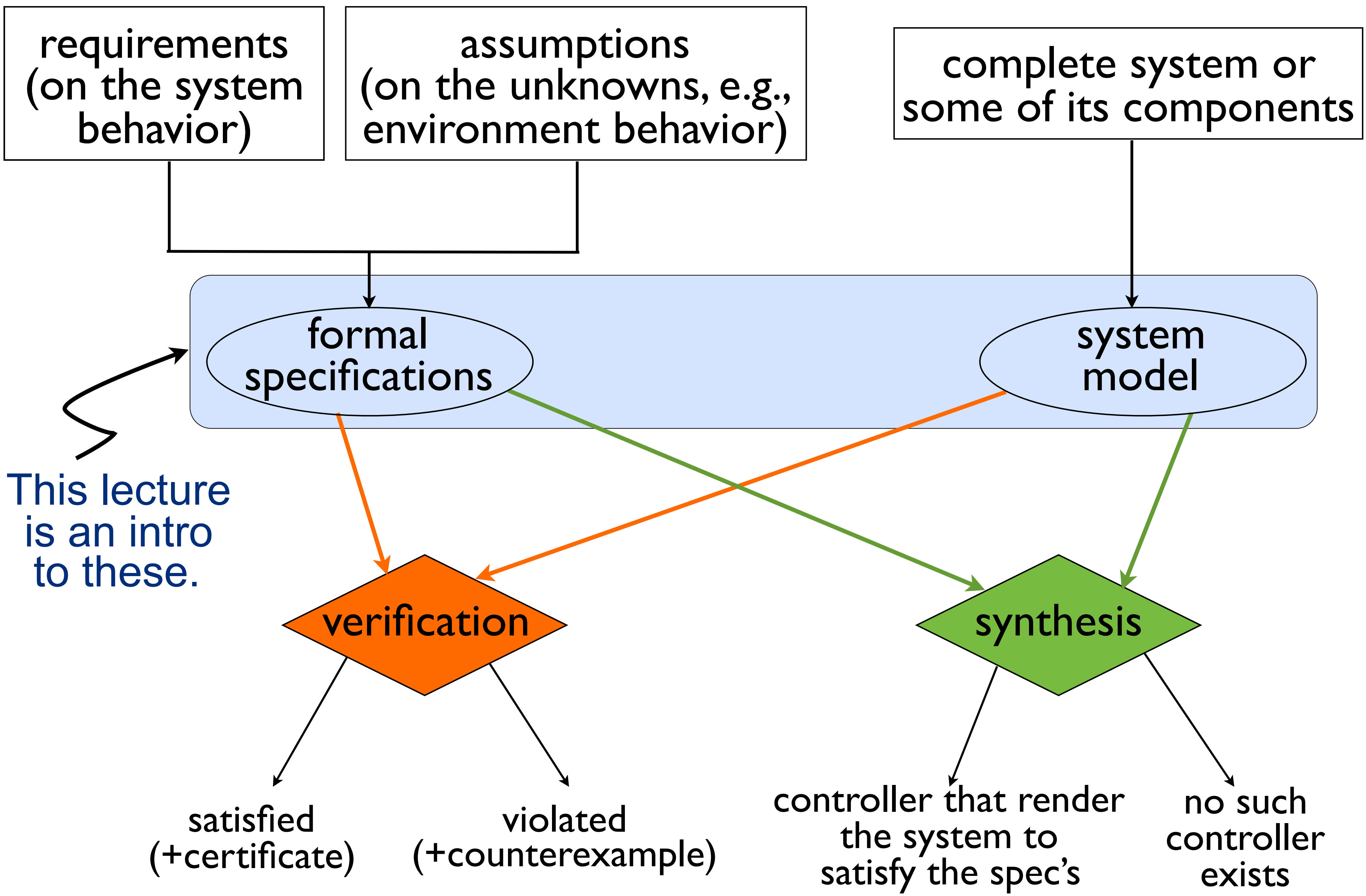
- Modeling (discrete) concurrent systems: transition systems, concurrency and interleaving
- Linear-time properties: invariants, safety and liveness properties



Principles of Model Checking,
C. Baier and J.-P. Katoen,
The MIT Press, 2008

Chapters 2.1, 2.2, 3.2-3.4

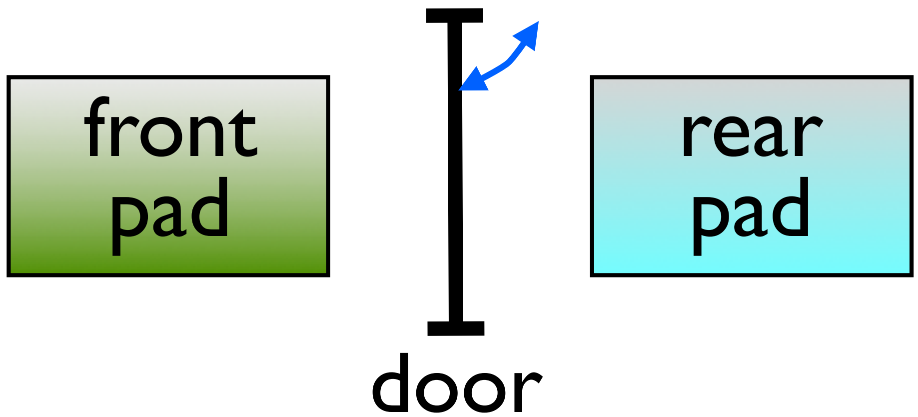
This short-course is on this picture applied to a particular class of systems/problems.

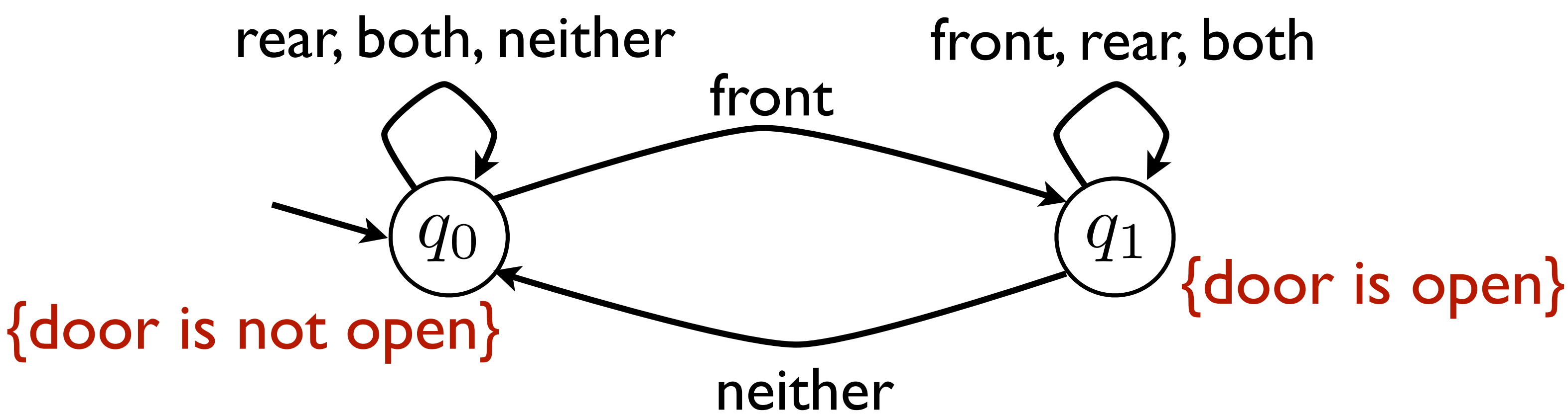


Finite transition system

A *finite transition system* is a mathematical description of the behavior of systems, plants, controllers or environments with finite (discrete)

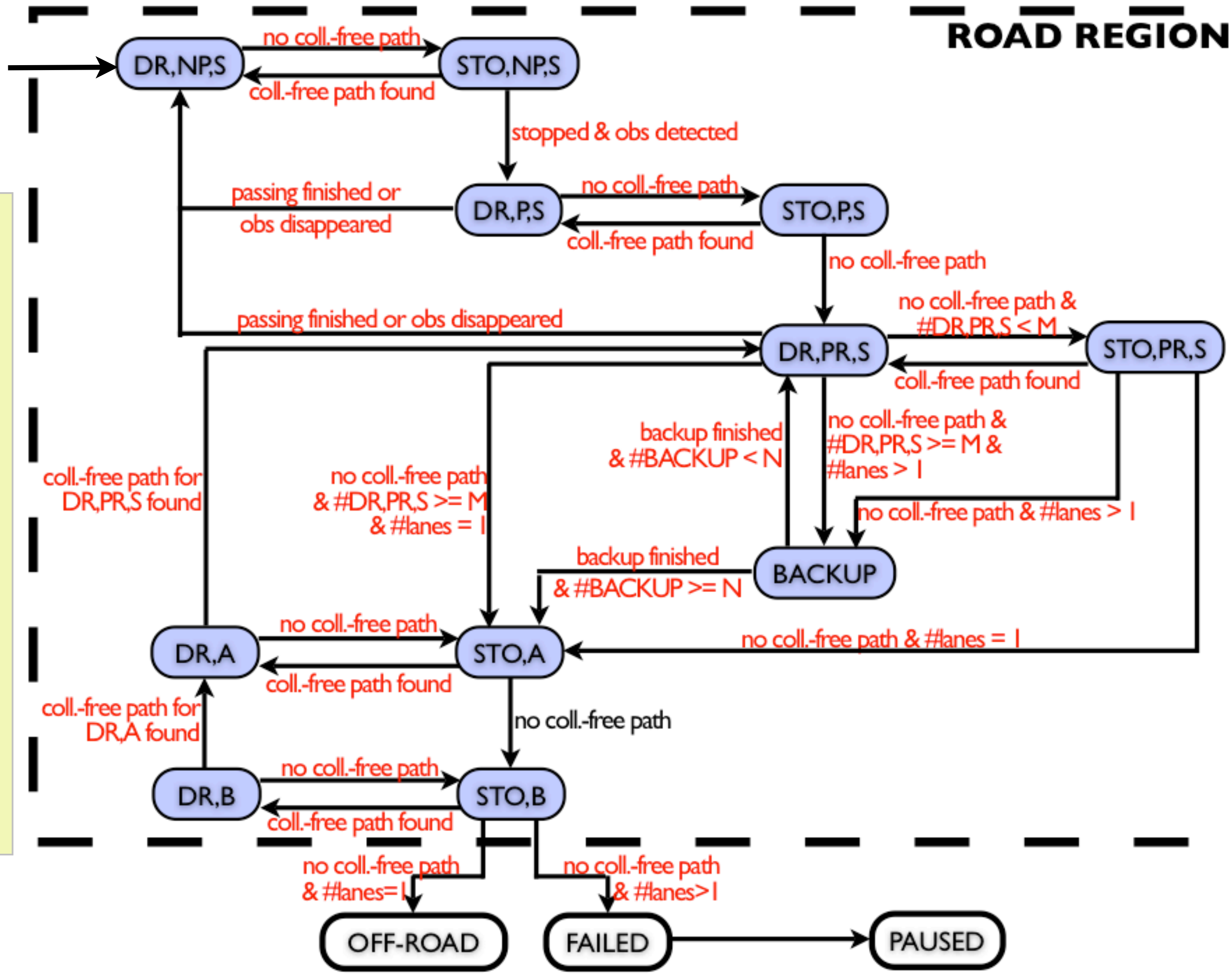
- inputs,
- outputs, and
- internal states and transitions between the states.





Finite transition system

Example: Traffic logic planner in Alice.

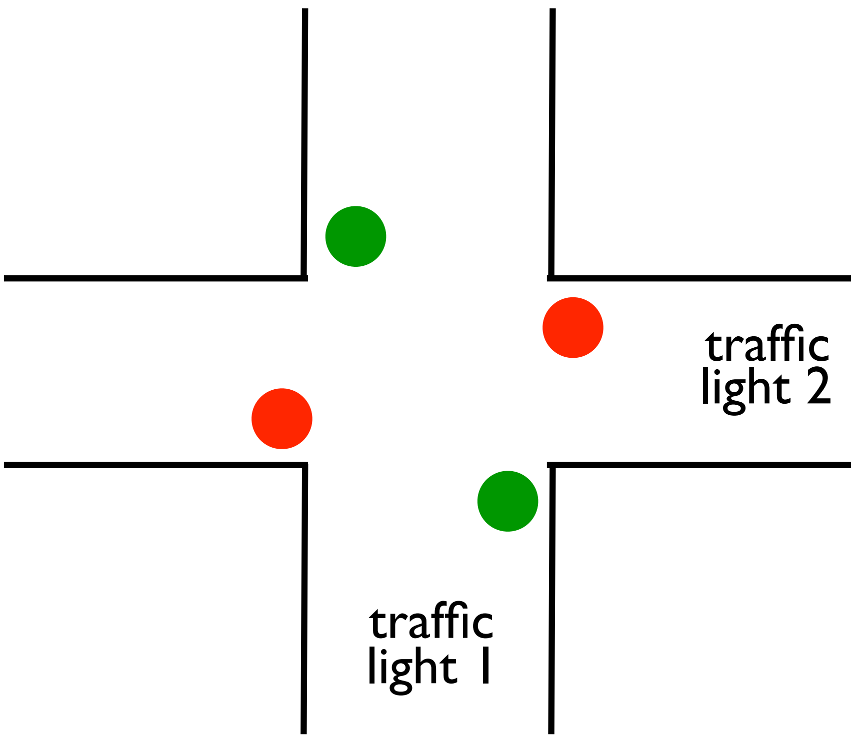


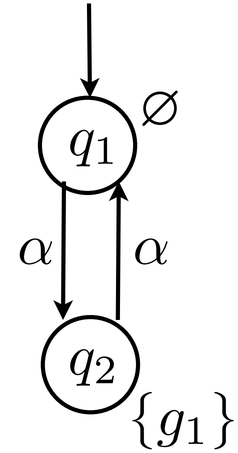
Partial nomenclature:

DR = drive.
STO = stop.
NP = no passing, no reversing.
P = passing, no reversing.
PR = passing, reversing allowed.
S = safe clearance with obstacle.
A = aggressive clearance with obstacle.
B = no clearance with obstacle.

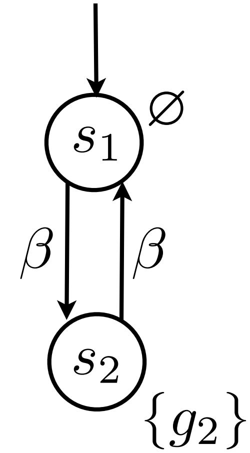
Finite transition system

Example: Traffic lights.

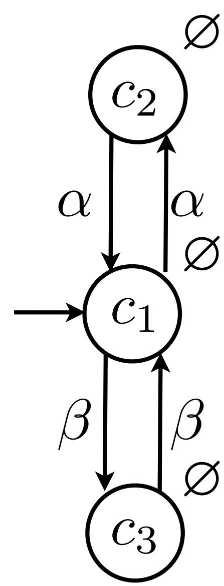




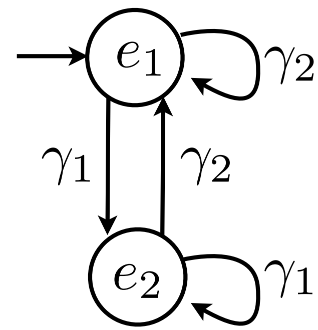
traffic
light 1



traffic
light 2



controller



environment

Preliminaries

A ***proposition*** is a statement that can be either true or false, but not both.

Examples:
- "Traffic light is green" is a proposition.
- "The front pad is occupied" is a proposition.
- "Is the front pad occupied?" is <u>not</u> a proposition.

An ***atomic proposition*** is one whose truth or falsity does not depend on the truth or falsity of any other proposition.

Examples:
- All propositions above are atomic propositions.
- "If traffic light is green, the car can drive" is <u>not</u> an atomic proposition.

For notational brevity, use propositional variables to abbreviate propositions. For example,

$$p \equiv \text{ Traffic light is green}$$

$$q \equiv \text{ Front pad is occupied}$$

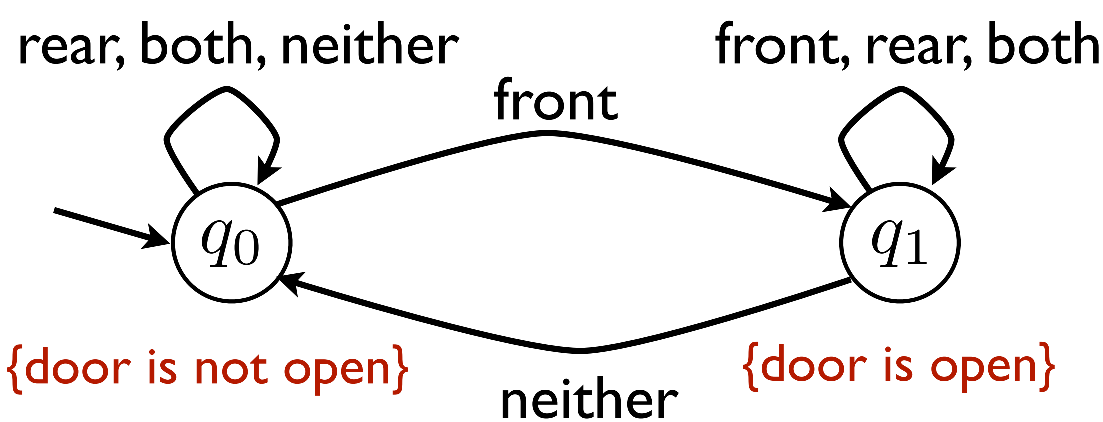
Finite transition system

A transition system TS is a tuple $TS = (S, Act, \rightarrow, I, AP, L)$, where

- S is a set of states,
- Act is a set of actions,
- $\rightarrow \subseteq S \times Act \times S$ is a transition relation,
- $I \subseteq S$ is a set of initial states,
- AP is a set of atomic propositions,
- $L : S \rightarrow 2^{AP}$ is a labeling function, and

TS is called finite if S, Act, and AP are finite.

- AP depends on the characteristics of the system of interest.
- For state s, $L(s)$ is the set of atomic propositions that are satisfied at s.
- Labels model outputs or observables.
- Actions model inputs or "communication."



example

$S = \{q_0, q_1\}$
$Act = \{rear, front, both, neither\}$
$\rightarrow = \{(q_0, front, q_1), (q_1, neither, q_0),$
$\quad (q_1, rear, q_1), \ldots\}$
$I = \{q_0\}$
$L(q_0) = \{door\ is\ not\ open\}$
$L(q_1) = \{door\ is\ open\}$

Propositional logic

From "Specifying Systems" by L. Lamport: Propositional logic is the math of the Boolean values, true and false, and the operators $\neg, \wedge, \vee, \rightarrow$

Given finite set AP of atomic propositions, the set of propositional logic formulas is inductively defined by:

- true is a formula;
- any $a \in AP$ is a formula;
- if ϕ_1, ϕ_2, and ϕ are formulas, so are $\neg\phi$ and $\phi_1 \wedge \phi_2$; and
- nothing else is a formula.

Notation

- Connectives:

$\neg$ (negation), $\wedge$ (and)

$\vee$ (or), $\rightarrow$ (implies)

- 1 for "true" and 0 for "false."

Example propositional logic formulas obtained by applying the above four rules:

$$\phi_1 \vee \phi_2 := \neg(\neg\phi_1 \wedge \neg\phi_2)$$

$$\phi_1 \rightarrow \phi_2 := \neg\phi_1 \vee \phi_2$$

The *evaluation function* $\mu : AP \rightarrow \{0, 1\}$ assigns a truth value to each $a \in AP$.

The truth value $\mu(\Phi)$ of a formula Φ is determined by substituting the values for the atomic propositions specified by μ.

Given: $AP = \{a, b, c\}$, $\mu(a) = 0$ and $\mu(b) = \mu(c) = 1$.

$$\Phi_1 = (a \wedge \neg b) \vee c, \quad \mu(\Phi_1) = 1$$

$$\Phi_2 = (a \wedge \neg b) \wedge c, \quad \mu(\Phi_2) = 0$$

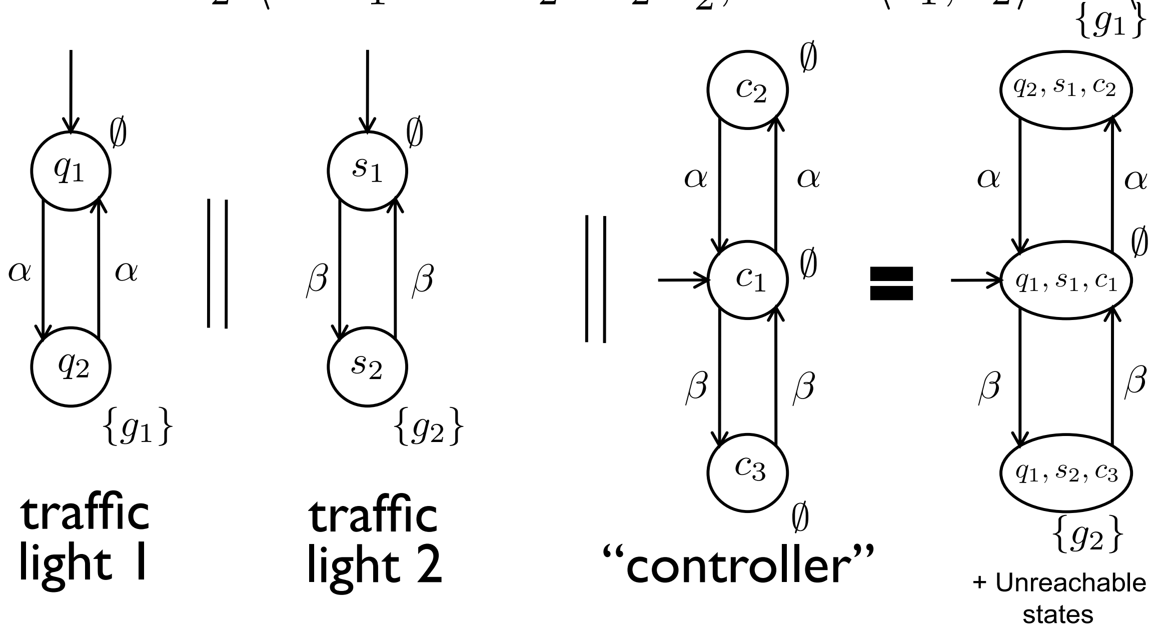
Composition of transition systems (by handshaking)

Let $TS_1 = (S_1, Act_1, \rightarrow_1, I_1, AP_1, L_1)$ and $TS_2 = (S_2, Act_2, \rightarrow_2, I_2, AP_2, L_2)$ be transition systems. Their parallel composition, $TS_1 || TS_2$ is the transition system defined by

$$TS_1 || TS_2 = (S_1 \times S_2, Act_1 \cup Act_2, \rightarrow, I_1 \times I_2, AP_1 \cup AP_2, L)$$

where $L(\langle s_1, s_2 \rangle) = L_1(s_1) \cup L_2(s_2)$ and $\rightarrow$ is defined by the following rules:

- If $\alpha \in Act_1 \cap Act_2$, $s_1 \xrightarrow{\alpha}_1 s_1'$, and $s_2 \xrightarrow{\alpha}_2 s_2'$, then $\langle s_1, s_2 \rangle \xrightarrow{\alpha} \langle s_1', s_2' \rangle$.
- If $\alpha \in Act_1 \setminus Act_2$ and $s_1 \xrightarrow{\alpha}_1 s_1'$, then $\langle s_1, s_2 \rangle \xrightarrow{\alpha} \langle s_1', s_2 \rangle$.
- If $\alpha \in Act_2 \setminus Act_1$ and $s_2 \xrightarrow{\alpha}_2 s_2'$, then $\langle s_1, s_2 \rangle \xrightarrow{\alpha} \langle s_1, s_2' \rangle$.

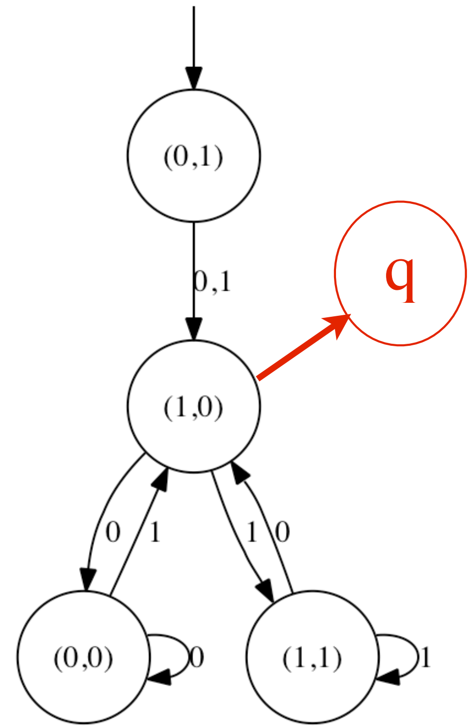


Paths of a finite transition system

Given a transition system $TS = (S, Act, \rightarrow, I, AP, L)$.
For $s \in S$,

$$Post(s) := \left\{ s' \in S : \exists a \in Act \text{ s.t. } s \xrightarrow{a} s' \right\}$$

- Example: $Post((0,0)) = \{(0,0),(1,0)\}$.
- A state s is *terminal* iff $Post(s)$ is empty.
- A sequence of states, either finite $\pi = s_0 s_1 s_2 \ldots s_n$ or infinite $\pi = s_0 s_1 s_2 \ldots$, is a *path fragment* if $s_{i+1} \in Post(s_i), \ \forall i \geq 0$.



- A *path* is a path fragment s.t. $s_0 \in I$ and it is
  - either finite with terminal s_n
  - or infinite.
- Denote the set of paths in *TS* by $Path(TS)$.

a path:

$$(0,1) \xrightarrow{,1} (1,0) \xrightarrow{1} (1,1) \xrightarrow{1} (1,1) \xrightarrow{0} \cdots$$

not a path:

$$(1,0) \xrightarrow{0} (0,0) \xrightarrow{0} (0,0) \xrightarrow{1} (1,0) \xrightarrow{0} \cdots$$

not a path:

$$(0,1) \xrightarrow{,1} (1,0) \xrightarrow{1} (1,1).$$

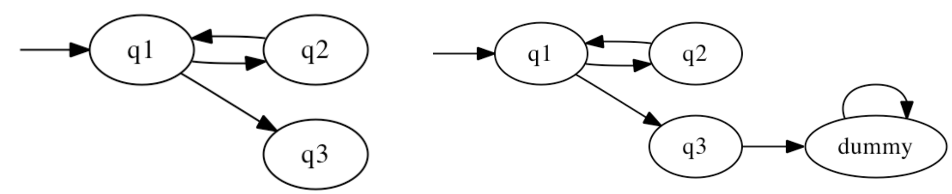
Traces of a finite transition system

Consider a finite transition system

$$TS = (S, Act, \rightarrow, I, AP, L)$$

with no terminal states (wlog).

Equivalent FSMs w/ and w/o terminal state



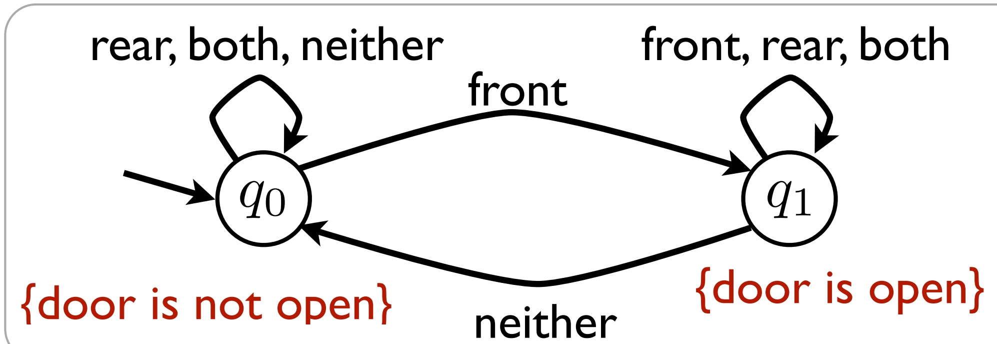
The *trace* of an infinite path fragment $\pi = s_0 s_1 s_2 \ldots$ is defined by

$$trace(\pi) = L(s_0)L(s_1)L(s_2)\ldots$$

sequence of sets of atomic propositions that are valid in the states along the path

The set, $Traces(TS)$, of traces of TS is defined by

$$Traces(TS) = \{trace(\pi) : \pi \in Paths(TS)\}$$



Actions: $f, f, n, b, f, f, b, \ldots$
Path: $q_0 q_1 q_1 q_0 q_0 q_1 q_1 q_1 \ldots$
Trace: $\neg o, o, o, \neg o, \neg o, o, o, o, \ldots$

(with some abuse of notation)

Linear-time properties

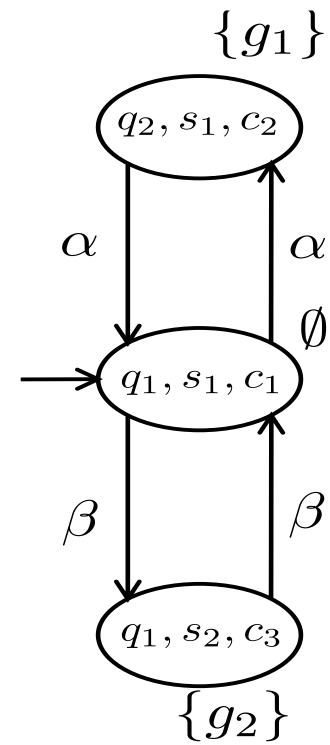
A linear-time (LT) property P over atomic propositions in AP is a set of infinite sequences over 2^{AP}.

Let P be an LT property over AP and $TS = (S, Act, \to, I, AP, L)$ be a transition system.

TS satisfies P, denoted as $TS \models P$, iff $Traces(TS) \subseteq P$.

traces of *TS*

admissible, desired, undesired, etc. behavior



<u>Example</u>: $AP = \{red1, green1, red2, green2\}$

P1 = "The first light is infinitely often green."

[$A_0 A_1 A_2 \dots$ with $green1 \in A_i \subseteq 2^{AP}$ holds for infinitely many i]

- $\surd$ $\{r1, g2\}\{g1, r2\}\{r1, g2\}\{g1, r2\} \dots$
- $\surd$ $\emptyset\{g1\}\emptyset\{g1\}\emptyset\{g1\}\emptyset \dots$
- $\surd$ $\{g1, g2\}\{g1, g2\}\{g1, g2\} \dots$
- $\times$ $\{r1, g2\}\{r1 g1\}\emptyset\emptyset \dots$

P2 = "The lights are never both green simultaneously."

[$A_0 A_1 A_2 \dots$ with $green1 \notin A_i$ or $green2 \notin A_i$, for all $i \geq 0$]

The transition system satisfies P2, but it does not satisfy P1.

Invariants

Notation: repeat infinitely many times

An LT property P_Φ over AP is an *invariant* with respect to a propositional logic formula Φ over AP if

$$P_\Phi = \{A_0A_1A_2\ldots \in (2^{AP})^\omega \ : \ A_j \models \Phi \ \forall j \geq 0\}.$$

For $A \subseteq AP$, let the evaluation μ_A be the characteristic function of A.

$A \models \Phi$ iff $\mu_A(\Phi) = 1$

<u>Example</u>: The LT property "the lights are never both green simultaneously" is an invariant with respect to $\Phi = \neg green1 \vee \neg green2$.

Given TS, Φ, and P_Φ, $TS \models P_\Phi$?

The following four statements are equivalent.

1. $TS \models P_\Phi$
2. $trace(\pi) \in P_\Phi, \ \forall \pi \in Path(TS)$
3. $L(s) \models \Phi, \ \forall s \in S$ on a path of TS
4. $L(s) \models \Phi, \ \forall s \in Reach(TS)$

A state s is reachable if there exists an execution fragment s.t. $s_0 \in I$ and

$$s_0 \xrightarrow{a_1} s_1 \xrightarrow{a_2} \cdots \xrightarrow{a_n} s_n = s$$

$Reach(TS)$: set of reachable states in TS

Invariants are state properties. That is, for verification, find the reachable states and check Φ.

Safety properties

An LT property P_{safe} is a *safety* property if for all words $\sigma \in (2^{AP})^\omega \backslash P_{safe}$ there exists a finite prefix $\hat{\sigma}$ of σ s.t.

$$P_{safe} \cap \{\sigma' \in (2^{AP})^\omega : \hat{\sigma} \text{ is a finite prefix of } \sigma'\} = \emptyset.$$

Bad things have happened in the bad prefix $\hat{\sigma}$. Hence, no infinite word that starts with $\hat{\sigma}$ satisfies P_{safe}.

Example: *AP* = {red, green, yellow}

- "At least one of the lights is always on" is a safety property.

$\{\sigma = A_0 A_1 \ldots \ : \ A_j \subseteq AP \wedge A_j \neq \emptyset\}$

Bad prefixes: finite words that contain $\emptyset$.

- "Two lights are never on at the same time" is a safety property.

$\{\sigma = A_0 A_1 \ldots \ : \ A_j \subseteq AP \wedge card(A_j) \leq 1\}$

Bad prefixes: finite words that contain {red,green}, {red,yellow}, and so on.

Any invariant is a safety property. There are safety properties that are not invariant.

Example: *AP* = {red, yellow}

"Each red is immediately preceded by a yellow" is a safety property, but not invariant (because it is not a state property).

Sample bad prefixes:

$\emptyset\emptyset\{r\}$

$\{y\}\{y\}\{r\}\{r\}\emptyset\{r\}$

Liveness properties

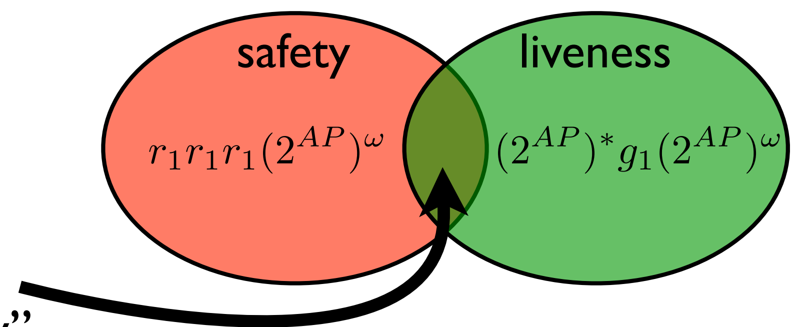
An LT property P is a liveness property if and only if for each finite word w of 2^{AP} there exists an infinite word $\sigma \in (2^{AP})^\omega$ satisfying w$\sigma \in P$.

<u>Example</u>: Two traffic lights with $AP = \{red1, green1, red2, green2\}$

- First light will *eventually* turn green
- First light will turn green *infinitely often*

<u>Use of liveness properties</u>:

- specify the absence of (undesired) infinite loops or progress toward a goal.
- rule out executions that cannot realistically occur (fairness), e.g., in an asynchronous execution, every process is activate infinitely often.



<u>Example:</u> Is the following a safety property? Liveness?

"the first light is eventually green
after it is initially red three time instances in a row"

<u>Answer:</u> It is a combination of a safety and a liveness property.

- Liveness: any finite word can be extended by an infinite word $A_0 A_1 A_2 \ldots$ with $green1 \in A_j$ for some $j \geq 0$.
- Safety: any finite word $A_0 A_1 A_2$ with $red1 \notin A_i$ for any $i \in \{0, 1, 2\}$ is a bad prefix.

Invariant	Safety	Liveness
state condition	something bad never happens	something good will happen eventually
violated at individual states	any infinite run violating the property has a finite prefix	violated only by infinite runs
verification: find the reachable states and check the invariant condition	verification: ?	verification: ?

Nondeterministic finite automaton (NFA)

A nondeterministic finite automaton $\mathcal{A} = (Q, \Sigma, \delta, Q_0, F)$ is a tuple with

- A is a set of states,
- Σ is an alphabet,
- $\delta : Q \times \Sigma \to 2^Q$ is a transition function,
- $Q_0 \subseteq Q$ is a set of initial states, and
- $F \subseteq Q$ is a set of accept (or: final) states.

set of finite words

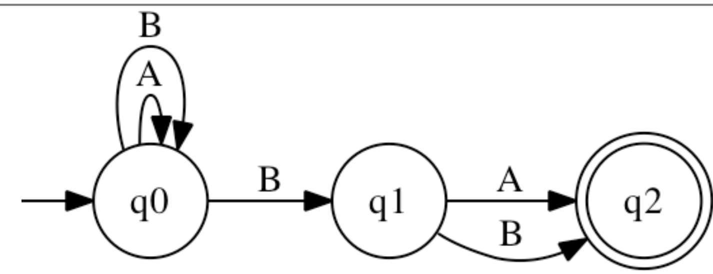
Let w $= A_1 \ldots A_n \in \Sigma^*$ be a finite word.
A *run* for w in $\mathcal{A}$ is a finite sequence of states $q_0 q_1 \ldots q_n$ s.t.

- $q_0 \in Q_0$
- $q_i \xrightarrow{A_{i+1}} q_{i+1}$ for all $0 \leq i < n$.

A run $q_0 q_1 \ldots q_n$ is called accepting if $q_n \in F$.

A finite word in accepted if it leads to an accepting run.

The *accepted language* $\mathcal{L}(\mathcal{A})$ of $\mathcal{A}$ is the set of finite words in Σ^* accepted by $\mathcal{A}$.



$Q = \{q0, q1, q2\}$, $\Sigma = \{A, B\}$

$Q_0 = \{q0\}$, $F = \{q2\}$

$\delta(q0, A) = \{q0\}$, $\delta(q0, B) = \{q0, q1\}$

$\delta(q1, A) = \{q2\}$, $\delta(q1, B) = \{q2\}$

$\delta(q2, A) = \emptyset$, $\delta(q0, B) = \emptyset$

word	run
empty word	q0
B	q0q1
ABA	q0q0q0q0
BBA	q0q0q0q0
BA	q0q1q2
BBA	q0q0q1q2

accepted

Regular safety properties

NFA: $\mathcal{A} = (Q, \Sigma, \delta, Q_0, F)$

A set $\mathcal{L} \subseteq \Sigma^*$ of finite strings is called a regular language if there is a nondeterministic finite automaton $\mathcal{A}$ s.t. $\mathcal{L} = \mathcal{L}(\mathcal{A})$.

language (set of finite words) accepted by the NFA

A safety property P_{safe} over AP is called *regular* if its set of bad prefixes constitutes a regular language over 2^{AP}.

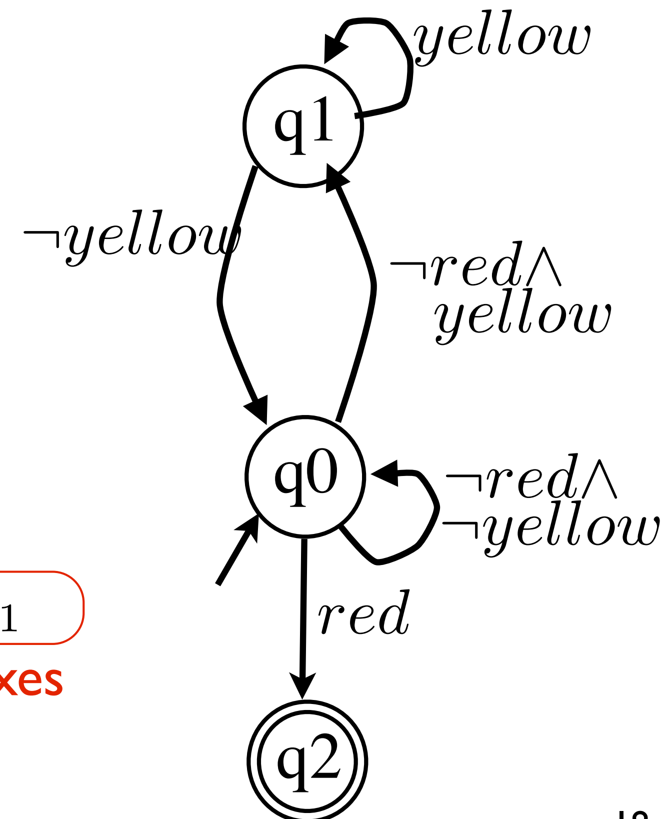
That is: $\exists$ NFA $\mathcal{A}$ s.t. $\mathcal{L}(\mathcal{A}) =$ bad prefixes of P_{safe}

Example: *AP* = {red, green, yellow}
"Each red must be preceded immediately by a yellow"
is a regular safety property.

Sample bad prefixes:

- {}{}{red}
- {}{red}
- {yellow}{yellow}{green}{red}
- $A_0 A_1 \ldots A_n$ s.t. $n > 0, red \in A_n,$ and $yellow \notin A_{n-1}$

general form of minimal bad prefixes



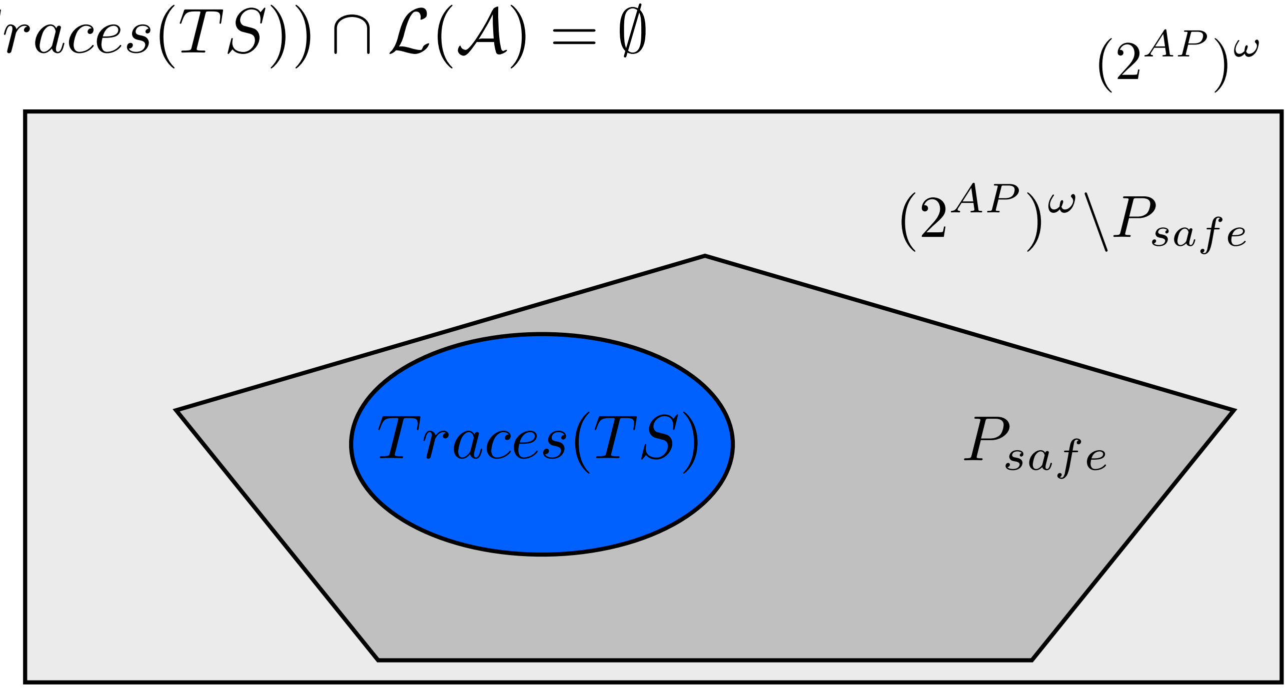
Verifying regular safety properties

Given a transition system TS and a regular safety property P_{safe}, both over the atomic propositions AP.

Let $\mathcal{A}$ be an NFA s.t. $\mathcal{L}(\mathcal{A}) = BadPref(P_{safe})$.

$$
\begin{aligned}
TS \models P_{safe} \quad & \text{iff} \quad Traces(TS) \subseteq P_{safe} \\
& \text{iff} \quad Traces(TS) \cap ((2^{AP})^\omega \setminus P_{safe}) = \emptyset \\
& \text{iff} \quad Traces(TS) \cap BadPref(P_{safe}).(2^{AP})^\omega = \emptyset \\
& \text{iff} \quad pref(Traces(TS)) \cap BadPref(P_{safe}) = \emptyset \\
& \text{iff} \quad pref(Traces(TS)) \cap \mathcal{L}(\mathcal{A}) = \emptyset
\end{aligned}
$$

finite prefixes



For words w and σ, w.σ denotes their concatenation.

Invariant	Safety	Liveness
state condition	something bad never happens	something good will happen eventually
violated at individual states	any infinite run violating the property has a finite prefix	violated only by infinite runs
verification: find the reachable states and check the invariant condition	verification: based on nondeterministic finite automaton which accepts "finite runs"	verification: ?

Nondeterministic Buchi automaton (NBA)

A nondeterministic Buchi automaton is same as an NFA $\mathcal{A} = (Q, \Sigma, \delta, Q_0, F)$ with its runs interpreted differently.

Let $w = A_1 A_2 \ldots \in \Sigma^\omega$ be an infinite string. A *run* for w in $\mathcal{A}$ is an infinite sequence $q_0 q_1 \ldots$ of states s.t.

- $q_0 \in Q_0$ and
- $q_0 \xrightarrow{A_1} q_1 \xrightarrow{A_2} q_2 \xrightarrow{A_3} \ldots$.

A run is *accepting* if $q_j \in F$ for infinitely many j.

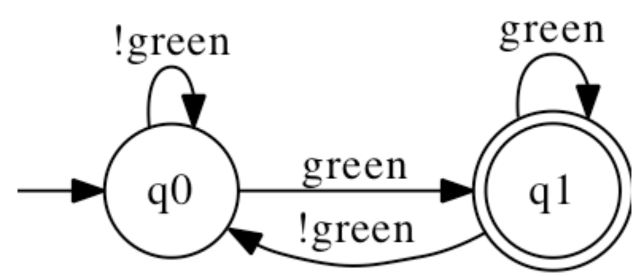
A string w is accepted by $\mathcal{A}$ if there is an accepting run of w in $\mathcal{A}$.

$\mathcal{L}_\omega(\mathcal{A})$: set of infinite strings accepted by $\mathcal{A}$.

A set of infinite string $\mathcal{L}_\omega \subseteq \Sigma^\omega$ is called an ω-regular language if there is an NBA $\mathcal{A}$ s.t. $\mathcal{L}_\omega = \mathcal{L}_\omega(\mathcal{A})$.

The NBA on the right accepts the infinite words satisfying the LT property: "infinitely often green."

AP = {red, green}



input word:

$\{green\}\{\}\{green\}\{\}\{green\}\{\}\ldots$

run:

$q_0 q_1 q_0 q_1 q_0 q_1 \ldots$

input word:

$(\{green, red\}\{\}\{green\}\{red\})^\omega$

run:

$q_0 q_1 q_0 q_1 q_0 q_1 \ldots$

NBA: $\mathcal{A} = (Q, \Sigma, \delta, Q_0, F)$

ω-Regular Properties

An LT property P over AP is called ω-regular if P is an ω-regular language over 2^{AP}.

Invariant, regular safety, and various liveness properties are ω-regular.

Let P be an ω-regular property and $\mathcal{A}$ be an NBA that represents the "bad traces" for P.

<u>Basic idea behind model checking ω-regular properties:</u>

$$
\begin{aligned}
TS \not\models P \quad & \text{if and only if} \quad Traces(TS) \not\subseteq P \\
& \text{if and only if} \quad Traces(TS) \cap \left((2^{AP})^\omega \setminus P \right) \neq \emptyset \\
& \text{if and only if} \quad Traces(TS) \cap \overline{P} \neq \emptyset \\
& \text{if and only if} \quad Traces(TS) \cap \mathcal{L}_\omega(\mathcal{A}) \neq \emptyset
\end{aligned}
$$

Invariant	Safety	Liveness
state condition	something bad never happens	something good will happen eventually
violated at individual states	any infinite run violating the property has a finite prefix	violated only by infinite runs
verification: find the reachable states and check the invariant condition	verification: based on nondeterministic finite automaton which accepts “finite runs”	verification: based on nondeterministic Buchi automaton which accepts infinite runs