NCS: Packet-Based Estimation and Control

From MurrayWiki

Prev: State Estimation | Course home | Next: Distributed Control

This lecture describes how to extend results in estimation and control to the case where the information between sensing, actuation and computation flows across a network with possible packet loss and time delay. We begin with the estimation problem, summarizing the results on Sinopoli et al on Kalman filtering with intermittent data, which uses average convergence as a stability metric. An alternative formulation is to use almost sure convergence, which gives improved results for lossy networks. Finally, we extend the results on estimation to the control setting, summarizing approaches in the cases where receipt of packets are acknowledge (TCP-like) or not acknowledged (UDP-like).

Contents

Outline

A. Kalman filtering with intermittent observations
  - Problem motivation and setup
  - Mathematical preliminaries (Jensen's inequality)
  - Main results: upper and lower bounds

B. Almost-sure state estimation with packet drops
  - Almost sure stability versus average stability
  - Performance versus data loss tradeoff

C. Packet-based control
  - TCP-like networks
  - UDP-like networks

D. Variable time-delays (if time) + anytime control?

Lecture Materials

- Lecture slides: Packet-Based Estimation and Control
- Lecture notes: NCS06, Chapter 5 - Packet-Based Estimation and Control

Additional Information

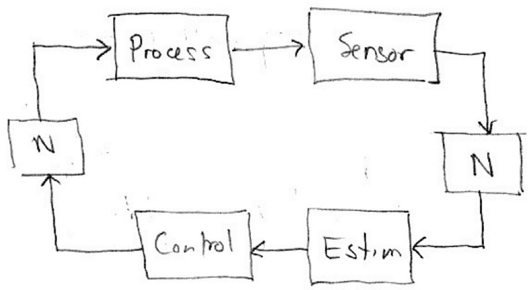
Further Reading

Missing: 1. Multi-description coding
2. Final comments

RMM 18 Mar 08

①

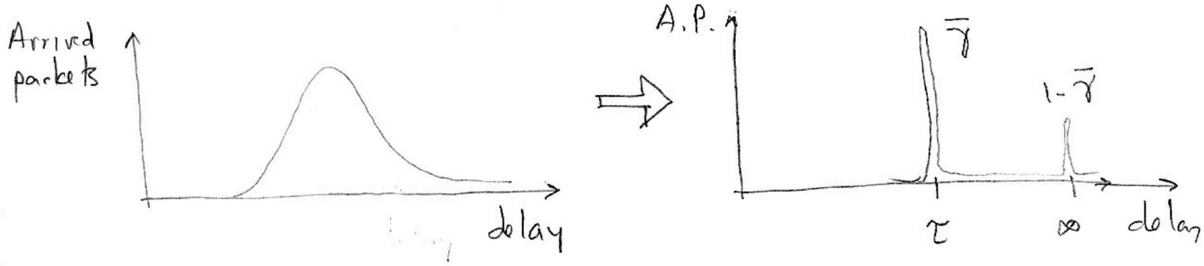
Problem setup



Network properties:

- packets delivered w/ prob $\bar{\gamma}$
- packets are timestamped
- ignore quantization

Delay model: for simplicity, assume packets arrive in time τ or are dropped



Questions:

1. How much network loss can we tolerate and maintain stability, performance, etc
2. If we process data before sending it across a link, what should we send?

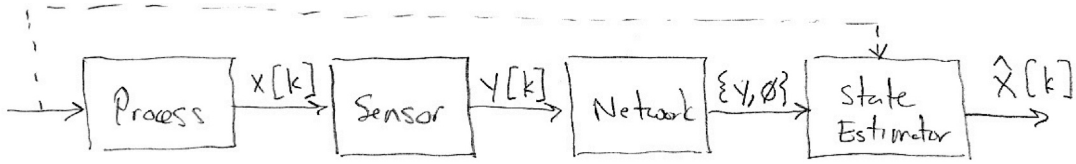
RMM experience

- Packet loss typically 10-20% (MVWT)
- Latency typically 60 ms ± 10 ms (MVWT) → use $\tau = 75$ms
- without spread, packets are often sparsely used

RMM 18 Mar 08

(2a)

Optimal Estimation w/ Intermittent Data



$$x[k+1] = Ax[k] + Fv[k] \qquad P(\gamma[k]=1) = \bar{\gamma}$$
$$y[k] = Cx[k] + w[k]$$

Network model:

$$\gamma[k] = \begin{cases} 1 & \text{packet arrived} \\ 0 & \text{packet dropped} \end{cases}$$

Bernoulli process w/ $P\{\gamma=1\} = \bar{\gamma}$
$\gamma[k], \gamma[j]$ independent

State estimator:

$$\hat{x}[k+1] = A\hat{x}[k] + \gamma[k] L[k] (y[k] - C\hat{x}[k])$$
$$L[k] = AP[k]C^T(R_w + CP[k]C^T)^{-1}$$
$$P[k+1] = AP[k]A^T + FR_vF^T - \gamma[k]APC^T(CPC^T + R_w)^{-1}CPA$$

MARE

Remarks

1. $P[k], \hat{x}[k]$ are <u>random processes</u> due to $\gamma[k]$
2. Can show this is the optimal estimator: let measurement covariance $= \sigma^2 I$ when $\gamma[k]=0$ and take $\sigma \to \infty$
3. Need to analyze $E\{P[k]\}$ to see if filter converges.
(Assume A is unstable from here on...)

RMM 18 Mar 08

2b

Properties of the modified algebraic Riccati equation (MARE)

$$g_\lambda(X) = AXA^T + \underbrace{F^T R_v F}_{Q} - \lambda AXC^T(CXC^T + \underbrace{R_w}_{R})^{-1} CXA^T$$

Define

$$\phi(K, X) = (1-\lambda)(AXA^T + Q) + \lambda(\tilde{F}X\tilde{F}^T + V)$$

$$F = A + KC \qquad V = Q + KRK^T$$

Properties:

(i) $K_X := -AXC^T(CXC^T + R)^{-1} \Rightarrow \phi(K_X, X) = g_\lambda(X)$

(ii) $g_\lambda(X) = \min_K \phi(K, X) \leq \phi(K, X) \quad \forall X$

* (iii) If $X \leq Y$ then $g_\lambda(X) \leq g_\lambda(Y)$

* (iv) For $\alpha \in [0,1]$ $\quad g_\lambda(\alpha X + (1-\alpha)Y) \geq \alpha g_\lambda(X) + (1-\alpha) g_\lambda(Y)$ [concave]

(v) $g_\lambda(X) \geq (1-\lambda) AXA^T + Q$

(vi) If $\bar{X} \geq g_\lambda(\bar{X})$ then $\bar{X} > 0$

* (vii) If X is a random variable, $(1-\lambda) A E\{X\} A^T + Q \leq E\{g_\lambda(X)\} \leq g_\lambda(E\{X\})$

(viii) If $\lambda_1 \leq \lambda_2$ then $g_{\lambda_1}(X) \leq g_{\lambda_2}(Y)$

Pf: Matrix algebra (see Sinopoli)

Try to get a difference equation for $E\{P[k]\}$:

$$E\{P[k+1]\} = A E\{P[k]\} A^T + F R_v F^T - \bar{\gamma} \underbrace{E\{ APC^T (CPC + R_w)^{-1} CPA\}}$$

Using Jensen's inequality ($E\{\phi(x)\} \le \phi(E\{x\})$ for convex* ϕ) and property (vii)

$$A E\{P[k]\} A^T \ge * \ge A E\{P[k]\} C^T \left(C E\{P[k]\} C^T + R_w \right)^{-1} C E\{P[k]\} A$$

We can thus bound $P[k]$ by two matrix difference equations

$$S[k] \le E\{P[k]\} \le V[k]$$

where

$$S[k+1] = (1-\bar{\gamma}) A S[k] A^T + F R_v F^T$$

$$V[k+1] = A V[k] A^T + F R_v F^T - \bar{\gamma} A V[k] C^T (C V[k] C^T + R_v)^{-1} C V[k] A$$

Convergence of lower bound: let $\rho = \max(\lambda(A))$. Effect of $(1-\bar{\gamma})$ is to reduce "gain" of $A \Rightarrow$ unstable if $(1-\bar{\gamma})\rho^2 > 1$

or

$$\bar{\gamma} \le \gamma_{min} := 1 - 1/\rho^2 \qquad \text{(unstable)}$$

This gives a strict lower bound on $\bar{\gamma}$ for stability of $E\{P[k]\}$

Convergence of upper bound: upper bound can be regarded as a modified algebraic Riccati equation (MARE)

$$g_\lambda(X) = AXA^T + \underbrace{F^T R_v F}_{Q} - \lambda AXC^T(CXC^T + R_w)^{-1} CXA^T$$

$$V[k+1] = g_{\bar\gamma}(V[k])$$

<u>Thm</u> (Sinopoli et al, 2004) If $(A, Q^{1/2})$ is controllable, (A, C) is detectable and A is unstable ($\rho(A) > 1$), then $\exists\ \gamma_c \in [0, 1)$ such that

1) $\lim_{k \to \infty} E\{P[k]\} = \infty$ for $0 \le \gamma \le \gamma_c$ and some $P[0] > 0$

2) $E\{P[k]\} \le M_{P[0]}$ for $\gamma_c < \gamma < 1$ and any $P[0] > 0$

where $M_{P[0]} > 0$.

<u>Remarks</u>

1. Can find bounds on γ_c via LMIs:

$$\phi(K, X) = (1-\gamma)(AXA^T + Q) + \gamma((A+KC)X(A\cdot KC^T) + KR_wK^T + Q)$$

$$\gamma_{max} = \arg\min_\gamma \left[(\hat{K}, \hat{X}) \mid \underbrace{\hat{X} > \phi(\hat{K}, \hat{X})} \right]$$

$\rightarrow \quad \gamma_{min} < \gamma_c < \gamma_{max}$ Sufficient condition for convergence (see Sinopoli et all, 2004)

2. Can compute γ_{max} and use this as a safe bound for $\bar\gamma$
3. If C is invertible then $\gamma_{min} = \gamma_c = \gamma_{max}$

RMM 18 Mar 08

5

Examples (slides)

Alternative formulation: probabalistic bounds

Motivation: convergence of $E\{P[k]\}$ can be too strong a condition

Example:

$$x[k+1] = a\,x[k] + v[k]$$

Suppose no packets received in k steps

$$a = 2 \Rightarrow \gamma_c = 0.74 < 1 - 1/a^2$$

$$E\{P[k]\} \geq (0.26^k) 4^k P_0 = 1.04^k P_0$$

If we let $k \to \infty$ then $E\{P[k]\} = \infty$. But very unlikely

Instead, try to find bound of the form $P[P[k] < M] = 1 - \varepsilon$ and plot M vs ε

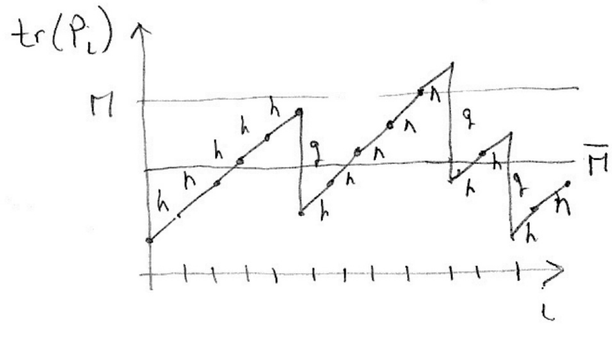
Thm: Assume packet arrival sequences are iid (eg. $\gamma[k]$ Bernoulli) and let $\bar{\gamma}$ be $E\{\gamma[k]\}$. If $\bar{\gamma} > 0$ then for any $0 < \varepsilon < 1$, there exists $M(\varepsilon) < \infty$ such that $P[k] < M(\varepsilon)$ with probability $1-\varepsilon$.

Idea behind proof: keep track of string of packets not received

$$h(X) = AXA^T + Q \qquad \text{no measurement}$$

$$g(X) = AXA^T + Q - AXC^T(CXC^T + R)^{-1}CXA^T \qquad \text{measurement}$$

Given a packet loss sequence $P[k] = h^{i_\ell} \circ g^{i_{\ell-1}} \circ \cdots \circ h^{i_2} \circ g^{i_1}$ where $\sum_{j=1}^{\ell} i_j = k$.



Show that if you ever get a packet, then $g(P[k]) < \bar{M}$.

Keep track of probability of getting ℓ dropped packets in a row a choose $M > h^\ell(\bar{M})$.

RMM 18 Mar 08

6b

<u>Proposition</u> Define $\lambda_h(X) = \mathrm{Tr}(h(X))/\mathrm{Tr}(X)$. Then

$$\lambda_h(X) \leq 1 + \rho(A^T A) =: \bar{\rho}$$

for all $X > 0$ such that $\mathrm{Tr}(X) \geq \mathrm{Tr}(Q)$, where $\rho(A^T A)$ is the largest eigenvalue of $A^T A$.

<u>Pf</u> Manipulation of trace

<u>Lemma</u> $\exists\ \bar{M} > 0$ such that for any $X > 0$, $h(X) \leq \bar{M}$

<u>Pf</u> (for C invertible). Let $f(X) = AXA^T - AXC^T(CXC^T + R)^{-1}CXA^T$.

Note that

$$f(t\, C^{-1}RC^{-T}) = tAC^{-1}RC^{-T}A^T - AC^{-1}R\cancel{C^{-T}C^T}(tR + R)^{-1}RC^{-T}A^T$$
$$= t(\qquad) - \frac{1}{t+1}(\qquad)$$
$$= \frac{t}{t+1}(AC^{-1}RC^{-T}A^T) \leq AC^{-1}RC^{-T}A$$

Now $h(X) = f(X) + Q$. For any $X > 0$, we can choose t such that $t\,C^{-1}RC^{-T} > X \Rightarrow f(X) + Q < f(tC^{-1}RC^{-T}) + Q \leq AC^{-1}RC^{-T}A + Q =: \bar{M}$

RMM 18 Mar 08

6c

<u>Proof of theorem</u> (Note: need to assume $\varepsilon < \bar{\gamma}$?)

WLOG, assume packet is not received at time k (otherwise $M_\varepsilon = \bar{M}$).

Let $\bar{\xi} = 1 - \bar{\gamma}$ and let $k' = \max\{ l : l \le k$ and $\gamma_l = 1\}$ = last packet recieved.

$\text{Prob}(k - k' = N) = \bar{\gamma}\,\bar{\xi}^N$. Define

$$M_0 = \text{Tr}(P_0) \qquad M_1 = \text{Tr}(\bar{M}) \qquad \alpha_N = \bar{\lambda}_h^{\,N} \quad (\text{max growth in trace})$$

Suppose $0 < \varepsilon \le \bar{\gamma}$. Choose N such that $\bar{\gamma}\,\bar{\xi}^N < \varepsilon \Rightarrow$ with probability $1-\varepsilon$, we will get no more than N dropped packets in a row.

Now compute worst case value of $P[k]$ with N dropped packets

$$P[k] \le \alpha_n \, \text{tr}(\bar{M}) I =: M(\varepsilon)$$

<u>Remarks</u>

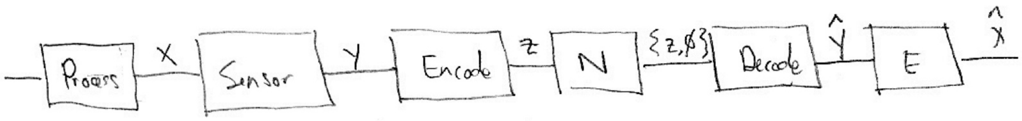
Check

1. Use the fact that $X = X^T$, $X > Y > 0 \Leftrightarrow \text{tr}(X) > \text{tr}(Y) > 0$
2. Bounds are conservative; can get better ones with work
3. Can extend to non-invertible C by sending multiple measurements

Roughly: send enough measurements in each packet to make $(C, CA, CA^2, \ldots)$ full rank

<u>Examples</u> (slides)

Compensating for packet loss



Goal: Find a way to encode data to minimize effect of packet loss

Approach # 1: send estimated state $\hat{x}$, decode via switched filter (predict/replace)

- Only possible if computation is available at sensor
- Often works well for signals with less than 1000 bytes (= packet size)
- Can show that this is optimal signal to send (Gupta) check

Approach # 2: send redudent packets - multi-description coding

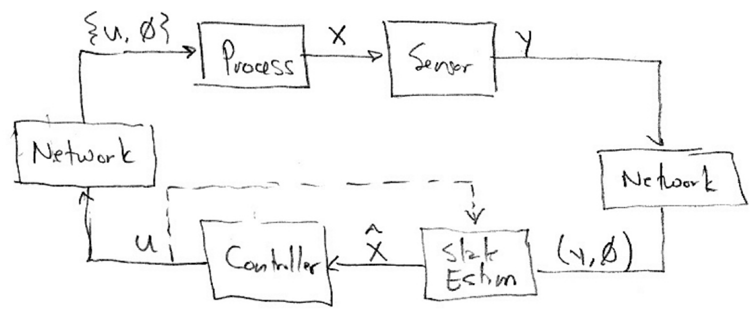
(briefly summarize Zhipu here) + Examples

RMM 18 Mar 08

8

Control across a lossy network

$p(\nu=1)=\bar{\nu}$ $P(\gamma=1)=\bar{\gamma}$



TCP-like: assume we know whether input u was applied

UDP-like: don't know if input was applied ⇒ state estimation is tricky

Approach # 1: Optimal control (LQG)

TCP-like:

- $\mathcal{I}_k = \{ y^k, \gamma^k, \nu^{k-1} \}$ $y^k = \{ y_k, y_{k-1}, \dots, y_1 \}, \dots$

  ⇒ assume we know at time k what has been received

- Can show separation principle holds: design optimal observer plus (time-varying) optimal controller. Controller is linear
- Stability if $\bar{\gamma} > \gamma_c$ and $\nu > \nu_c$

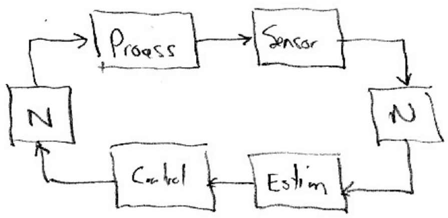
$$1 - \frac{1}{\max |\lambda_i^u(A)|} \le \nu_c \le 1 - \frac{1}{\prod |\lambda_i^u(A)|} \qquad \lambda_i^u = \text{unstable evalues}$$

UDP-like:

- $I_k = \{ y^k, \gamma^k \}$ Don't get ν^k
- No separation principle, get nonlinear fbk
- If $C = I$, necessary condition for stability is

$$\rho^2(A)\,(\bar{\gamma} - \bar{\nu} - 2\bar{\gamma}\bar{\nu}) < \bar{\gamma} + \bar{\nu} - 2\bar{\gamma}\bar{\nu}$$

Final comments & variations



RMM experience

- Packet drop for actuation is usually not a problem (low data flow)
- When possible send estimates & decode with predictors

Example (slides): control sending estimated state.