Homework problems for this week:

1. Prove the following:
   (a) The distribution law:
   \[ X \lor (Y \land Z) \equiv (X \lor Y) \land (X \lor Z) \].
   (b) The absorption law:
   \[ X \land (X \lor Y) \equiv X \].

2. Prove Theorem 5 in Sivilotti: \( X \implies Y \equiv \neg X \lor Y \).

3. The predicate \( x \neq y \) holds if exactly one of the predicates \( x \) and \( y \) is true. The proof that \( \neq \) is commutative is straightforward.
   (a) Show that \( \neq \) is also associative:
   \( (x \neq y) \neq z \equiv x \neq (y \neq z) \)
   (b) Consider a predicate \( X \) defined as \( x[0] \neq x[1] \neq \ldots \neq x[n] \). Which of the following statements are true?
      i. \( X \) is true if and only if the number of predicates \( x[i] \) that evaluate to true in the expression is an even number.
      ii. \( X \) is true if and only if the number of predicates \( x[i] \) that evaluate to false in the expression is an even number.
      iii. \( X \) is true if and only if the number of predicates \( x[i] \) that evaluate to true in the expression is an odd number.
iv. $X$ is true if and only if the number of predicates $x[i]$ that evaluate to false in the expression is an odd number.

v. None of the above.

(c) Prove your answer to problem 3b.

4. For a set $X$, the notation

$$\forall x \in X : p(x) : q(x)$$

stands for

for all $x$ in $X$ where $p(x)$ holds, $q(x)$ is true

or, equivalently,

for all $x$ in $X$, $p(x) \Rightarrow q(x)$,

where $\Rightarrow$ is implication. The set $X$ is often evident from the context and is left out of the formula, as in:

$$\forall x : p(x) : q(x),$$

where, for example, the set $X$ is the state of a system, or numbers, or the set of students at Caltech, depending on the context of the statement.

Note: In the course textbook, $p(x)$ is written as $p.x$, but we use the functional form to emphasize that $p$ is a predicate—a function from some space to the Boolean constants.

Likewise

$$\exists x \in X : p(x) : q(x)$$

stands for

there exists an (i.e. at least one) $x$ in $X$ where $p(x)$ holds for which $q(x)$ is true

or equivalently

there exists an $x$ in $X$ for which $p(x)$ and $q(x)$ hold,

which we write mathematically as

$$\exists x \in X : p(x) \land q(x)$$

Again, when the set $X$ is obvious from the context we write:

$$\exists x : p(x) : q(x)$$

(a) Is the following statement true?

$$\neg(\forall x : p(x) : q(x)) \equiv (\exists x : p(x) : \neg q(x))$$

(b) Prove your answer for a finite set $X$ or give a counter example. Recall that if $X$ is $\{x_0, \ldots, x_n\}$ then:

$$\forall x \in X : r(x) \equiv r(x_0) \land r(x_1) \land \ldots \land r(x_n)$$

$$\exists x \in X : r(x) \equiv r(x_0) \lor r(x_1) \lor \ldots \lor r(x_n)$$
5. Consider the following program:

\begin{verbatim}
Program Sivilotti-2.5.2
var b : Boolean
     n : \{0, 1, 2, 3\}
initially n = 1
assign
    b = 1 → n := n + 1
    n = 0 → b := false
\end{verbatim}

The addition in the first assignment is performed modulo 4.

(a) Draw a directed graph that represents this program. Make sure to label the initial nodes and label all actions using the action numbers above.

(b) Show that under the assumption of week fairness there is no guarantee that this program reaches a fixed point. Is there a fixed point under the assumption of strong fairness? If so, what is it?

(c) Does this program terminate?