Problem Session #1 for CS 142
Fall 2019

Tung Phan

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Booleans

- $\mathbb{B} = \{\textbf{true}, \textbf{false}\}$
- operations on Booleans: $\wedge, \vee, \neg, \equiv, \Rightarrow$ etc.
  - $\Rightarrow: \mathbb{B} \times \mathbb{B} \to \mathbb{B}$

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Predicates

- S is any set of interest
- predicates: $\mathcal{P} = \{P \mid P : S \to \mathbb{B}\}$
- operations on predicates: $\wedge, \vee, \neg, \equiv, \Rightarrow$ etc.
  - $\Rightarrow$: $\mathcal{P} \times \mathcal{P} \to \mathcal{P}$

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Everywhere brackets - implicit quantification

- $S = \mathbb{N}$

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Everywhere brackets - implicit quantification

- $S = \mathbb{N}$
- $\text{is\_even} \wedge \text{is\_divisible\_by\_2}$

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Everywhere brackets - implicit quantification

- $S = \mathbb{N}$
- is_even $\wedge$ is_divisible_by_2
- is_even $\equiv$ is_divisible_by_2

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Everywhere brackets - implicit quantification

- $S = \mathbb{N}$
- is_even $\wedge$ is_divisible_by_2
- is_even $\equiv$ is_divisible_by_2
- [is_even $\equiv$ is_divisible_by_2], for every number in $\mathbb{N}$, being even is the same as being divisible by 2

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Equivals

There can be many equivals on the same line, but this does not indicate chaining!

$$\textbf{false} \equiv \textbf{false} \equiv \textbf{true}$$

is not the same as

$$\textbf{false} \equiv \textbf{false} \wedge \textbf{false} \equiv \textbf{true}$$

On $\mathbb{B}$, $\equiv$ is clearly associative and commutative. What about $\mathcal{P}$?

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Equival axioms

1. $[((X \equiv Y) \equiv Z) \equiv (X \equiv (Y \equiv Z))]$ (associativity)

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Equival axioms

1. $[((X \equiv Y) \equiv Z) \equiv (X \equiv (Y \equiv Z))]$ (associativity)
2. $[X \equiv Y \equiv Y \equiv X]$ (commutativity)

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Equival axioms

1. $[((X \equiv Y) \equiv Z) \equiv (X \equiv (Y \equiv Z))]$ (associativity)
2. $[X \equiv Y \equiv Y \equiv X]$ (commutativity)
3. $[X \equiv X \equiv \textbf{true}]$ (definition of **true**)

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Disjunction axioms

1. $[X \vee (Y \vee Z) \equiv (X \vee Y) \vee Z]$ (associativity)

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Disjunction axioms

1. $[X \vee (Y \vee Z) \equiv (X \vee Y) \vee Z]$ (associativity)
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Disjunction axioms

1. $[X \vee (Y \vee Z) \equiv (X \vee Y) \vee Z]$ (associativity)
2. $[X \vee Y \equiv Y \vee X]$ (commutativity)
3. $[X \vee X \equiv X]$ (idempotence)

Disjunction axioms

1. $[X \vee (Y \vee Z) \equiv (X \vee Y) \vee Z]$ (associativity)
2. $[X \vee Y \equiv Y \vee X]$ (commutativity)
3. $[X \vee X \equiv X]$ (idempotence)
4. $[X \vee (Y \equiv Z) \equiv (X \vee Y) \equiv (X \vee Z)]$ (distribution over $\equiv$)

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Proof format

Prove $[A \equiv C]$:

$$
\begin{array}{ll}
 & A \\
\equiv & \{\text{why } [A \equiv B]\} \\
 & B \\
\equiv & \{\text{why } [B \equiv C]\} \\
 & C
\end{array}
$$

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Proof format

Prove $[A \equiv C]$:

$$
\begin{array}{ll}
 & A \equiv C \\
\equiv & \{\text{why } [A \equiv C \equiv D]\} \\
 & D \\
\equiv & \{\text{why } [D \equiv \textbf{true}]\} \\
 & \textbf{true}
\end{array}
$$

can also replace **true** by an axiom or theorem

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Theorem

$$[X \vee \textbf{true} \equiv \textbf{true}]$$

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Conjunction and implication

1. $[X \vee Y \equiv X \equiv Y \equiv X \wedge Y]$ (definition of $\wedge$ or "golden rule")

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Conjunction and implication

1. $[X \vee Y \equiv X \equiv Y \equiv X \wedge Y]$ (definition of $\wedge$ or "golden rule")
2. $[X \vee Y \equiv Y \equiv X \Rightarrow Y]$ (definition of $\Rightarrow$)

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Theorems

- $[X \wedge \textbf{true} \equiv X]$
- $[(X \wedge Y) \wedge Z \equiv X \wedge (Y \wedge Z)]$

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Negation and false axioms

1. $[X \vee \neg X]$ (law of excluded middle)

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Negation and false axioms

1. $[X \vee \neg X]$ (law of excluded middle)
2. $[\neg(X \equiv Y) \equiv X \equiv \neg Y]$ (distribution over $\neg$)

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Negation and false axioms

1. $[X \vee \neg X]$ (law of excluded middle)
2. $[\neg(X \equiv Y) \equiv X \equiv \neg Y]$ (distribution over $\neg$)
3. $[\textbf{false} \equiv \neg\textbf{true}]$ (definition of **false**)

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Theorems

- $[\neg\neg X \equiv X]$ (involutive property)
- $[X \Rightarrow Y \equiv \neg X \vee Y]$ (homework)

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Discrepance axiom

1. $[\neg(X \equiv Y) \equiv X \not\equiv Y]$ (definition of discrepance)

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Theorems

- $[(X \not\equiv Y) \equiv (Y \not\equiv X)]$ (commutativity)
- $[(X \not\equiv (Y \not\equiv Z)) \equiv ((X \not\equiv Y) \not\equiv Z)]$ (associativity)
- $[(X \equiv (Y \not\equiv Z)) \equiv ((X \equiv Y) \not\equiv Z)]$ (mutual associativity with $\equiv$)
- $[(X \not\equiv Y \not\equiv Z) \equiv (X \equiv Y \equiv Z)]$

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Quantification

- $(\mathbf{Q}i : r.i : t.i)$

Quantification

- $(\underbrace{\mathbf{Q}}_{\text{operator}} \underbrace{i}_{\text{bound variable}} : \underbrace{r}_{\text{range predicate}} .i : \underbrace{t}_{\text{term expression}} .i)$
- $\mathbf{Q}$ is binary, associative, symmetric and has an identity u
- if $\{i_0, i_1, \cdots, i_n\}$ is the set of values of i for which $r.i$ holds, then $(\mathbf{Q} i : r.i : t.i) = u \mathbf{Q} t.i_0 \mathbf{Q} t.i_1 \mathbf{Q} \ldots \mathbf{Q} t.i_n$
- some symbols

binary operator	identity	quantification version
$\wedge$	**true**	$\forall$
$\vee$	**false**	$\exists$
$+$	0	$\sum$
$\times$	1	$\prod$
max	$\perp$	**Max**
[]	skip	[]

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Quantification axioms

1. $(\mathbf{Q}i : \mathbf{true} : t.i) = (\mathbf{Q}i :: t.i)$ (short-hand for full range)
2. $(\mathbf{Q}i : \mathbf{false} : t.i) = u$ (empty range)
3. $(\mathbf{Q}i : i = E : t.i) = t.E$ (one-point rule)

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Examples of quantification

- $(\times i : 0 \leq i \leq 10 : i) = ?$

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Examples of quantification

- $(\times i : 0 \leq i \leq 10 : i) = ?$
- $(+i : \textbf{false} : i) = ?$

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Examples of quantification

- $(\times i : 0 \leq i \leq 10 : i) = ?$
- $(+i : \textbf{false} : i) = ?$
- $(\forall x : 1 < x \wedge x^2 < x : \textbf{false}) = ?$

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