CALIFORNIA INSTITUTE OF TECHNOLOGY
Control and Dynamical Systems

CDS 202
Problem Set #6

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Reading: Boothby, III.6–III.8, IV.6
Optional: AMR, Ch 5

1. Let G be a group and a manifold and assume that multiplication between group elements is a smooth operation. Prove that inversion is also smooth and hence G is a Lie group.

2. [Boothby, page 95, #6]

   (a) Let $G = SO(n) \times \mathbb{R}^n$ and define a product on G by $(A, v)(B, w) = (AB, Aw + v)$. Prove that G is a Lie group and identify the identity element and inverse for G.

   (b) Show that $SO(n) \times 0$ is a closed submanifold and subgroup of G.

3. [Boothby, page 151, #4]
   Find the one-parameter subgroups of $GL(2, \mathbb{R})$ generated by
   $$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$
   Find the corresponding actions on $\mathbb{R}^2$ and their infinitesimal generators, starting from the natural action of $GL(2, \mathbb{R})$ on $\mathbb{R}^2$.

4. [Boothby, page 151, #6]
   Prove that if A is a nonsingular $n \times n$ matrix and $X \in \mathbb{R}^{n\times n}$ then $Ae^X A^{-1} = \exp(AXA^{-1})$. From this deduce that $\det e^X = e^{\operatorname{tr} X}$. Use this to determine those matrices A such that e^{At}, $t \in \mathbb{R}$, is a one-parameter subgroup of $Sl(n, \mathbb{R})$.

5. [Warner, page 135, #16]
   Let G be a Lie group. Show that the set of right invariant vector fields on G forms a Lie algebra under the Lie bracket operation and that it is naturally isomorphic to T_eG.

6. [Warner, page 135, #16, cont.]
   Let $\phi : G \to G$ be the diffeomorphism defined by $\phi(g) = g^{-1}$. Prove that if $X \subset TG$ is a left invariant vector field on G then $\phi_*(X)$ is a right invariant vector field whose value at e is $-X(e)$. Further show that $X \mapsto \phi_*(X)$ gives a Lie algebra isomorphism of the Lie algebra of left invariant vector fields with the Lie algebra of right invariant vector fields on G.

   (A Lie algebra isomorphism is a linear mapping $A : V \to V$ which preserves the Lie bracket: $A[\xi, \eta] = [A\xi, A\eta]$.)