CALIFORNIA INSTITUTE OF TECHNOLOGY
Control and Dynamical Systems

CDS 202
Problem Set #2

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Reading:

Boothby, Chapter I and Sections III.1–III.3

Note: Most of the problems are taken from the exercises in Guillemin and Pollack. If you read Guillemin and Pollack, be warned that they treat manifolds slightly differently, using parameterizations instead of coordinate charts.

Problems:

1. [Guillemin and Pollack, page 5, #3]
   Let M, N, and P be smooth manifolds and let $f : M \to N$ and $g : N \to P$ be smooth maps.

   (a) Show that the composite map $g \circ f : M \to P$ is smooth.

   (b) Show that if f and g are diffeomorphisms, so is $g \circ f$.

   (You may use the fact that the composition of smooth functions between open subsets of Euclidian spaces are smooth.)

2. [Boothby II.1.2] Using stereographic projection from the north pole $N(0, 0, +1)$ of all of the standard unit sphere in $\mathbb{R}^3$ except $(0, 0, +1)$ determine a coordinate neighborhood U_N, ϕ_N. In the same way determine by projection from the south pole $S(0, 0, -1)$ a neighborhood U_S, ϕ_S (see figure in Boothby). Show that these two neighborhoods determine a C^∞ structure on S^2. Generalize to S^{n-1}.

3. [Guillemin and Pollack, page 6, #17]
   The *graph* of a map $f : M \to N$ is the subset of $M \times N$ defined by

   $$\operatorname{graph}(f) = \{(p, f(p)) : p \in M\}.$$

   Define $F : M \to \operatorname{graph}(f)$ by $F(p) = (p, f(p))$. Show that if f is smooth, F is a diffeomorphism; thus $\operatorname{graph}(f)$ is a manifold if M is.

4. [Guillemin and Pollack, page 6, #18]

   (a) An extremely useful function $f : \mathbb{R} \to \mathbb{R}$ is

   $$f(x) = \begin{cases} e^{-1/x^2} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

   Prove that f is smooth.

(b) Show that $g(x) = f(x-a)f(b-x)$ is a smooth function, positive on (a,b) and zero elsewhere. (Here $a < b$.) Then

$$h(x) = \frac{\displaystyle\int_{-\infty}^{x} g\,dx}{\displaystyle\int_{-\infty}^{\infty} g\,dx}$$

is a smooth function satisfying $h(x) = 0$ for $x < a$, $h(x) = 1$ for $x > b$ and $0 < h(x) < 1$ for $x \in (a,b)$.

h is called a *bump function.*

5. [Guillemin and Pollack, page 12, #4]
Prove that if $f : M \to N$ is a diffeomorphism then at each point p, df_p is an isomorphism between tangent spaces.