Problems:

1. Consider the following vector fields on \( \mathbb{R}^3 \):

\[
X(x) = \frac{\partial}{\partial x_2} - x_1 \frac{\partial}{\partial x_3}, \quad Y(x) = \frac{\partial}{\partial x_1}.
\]

Let \( x_0 = (0, 0, 0) \). Show that \( \phi^{-Y}_{h} \circ \phi^{-X}_{h} \circ \phi^{X}_{h}(x_0) = h^2 \phi^{[X,Y]}(x_0) \).

2. Show that if \( \Delta \) is a distribution of the form

\[
\Delta = \text{span}\{X_1, \ldots, X_d\}
\]

and we have \([X_i, X_j] \in \Delta \) for all \( i, j \) then for any \( X, Y \in \Delta, [X, Y] \in \Delta \). That is, to check involutivity of a distribution, we need only check that the pairwise brackets between basis elements lie in the distribution.

3. [Boothby, page 164, #4] Let \( N \subset M \) be a submanifold and let \( X, Y \in \mathcal{X}(M) \) be vector fields such that \( X_p, Y_p \in T_pN \) for \( p \in N \). Show that \( [X,Y]_p \in T_pN \) for all \( p \in N \).

4. [Boothby, page 164, #5] Let \( F : M \to N \) be a smooth submersion of \( M \) onto \( N \). Show that \( F^{-1}(q) \) for all \( q \in N \) are the leaves of a foliation on \( M \).

5. Consider the motion of a disk rolling on the plane, as shown below:

We can represent the configuration of the disk by the \( xy \) location of the disk and the angle of the disk with respect to a fixed line on the plane. We ignore the angle through which the disk rolls. We model the motion of the disk using a one vector field to represent the drive input (rolling) and another vector field to represent the steer input (twisting).

Let \( M = \mathbb{R}^2 \times S^1 \) and let \( p = (x, y, \theta) \) represent a point on \( M \). Consider the two vector fields

\[
X(p) = \cos \theta \frac{\partial}{\partial x} + \sin \theta \frac{\partial}{\partial y}, \quad Y(p) = \frac{\partial}{\partial \theta}
\]

(\( X \) is the drive vector field and \( Y \) is the steer vector field.)

(a) Compute the flows of \( X \) and \( Y \). Are they complete?
(b) Verify that $X$ and $Y$ are invariant under their own flows.

(c) Compute the Lie bracket between $X$ and $Y$. Show that the tangent vectors $X_p$, $Y_p$, and $[X,Y]_p$ span $T_pM$ for all $p \in M$.

(d) Consider the change of coordinates $z = \phi(x)$ given by

$$
\begin{bmatrix}
  z_1 \\
  z_2 \\
  z_3
\end{bmatrix} =
\begin{bmatrix}
  x \cos \theta + y \sin \theta \\
  x \sin \theta - y \cos \theta \\
  \theta
\end{bmatrix},
$$

(The new coordinates have the physical interpretation of being the origin of the spatial reference frame when viewed from a coordinate frame attached to the disk). Using the pushforward map for $\phi : \mathbb{R}^3 \to \mathbb{R}^3$, compute the $X$ and $Y$ vector fields in this new set of coordinates. (Hint: the final answer has a pretty simple form.)

(e) Show that $\phi_*[X,Y] = [\phi_* X, \phi_* Y]$ by explicit calculation.

(f) Show that $X$ and $Y$ are invariant under the group of rigid motions on $\mathbb{R}^2$, given by mappings

$$
F(x, y, \theta) =
\begin{bmatrix}
  x \cos \alpha - y \sin \alpha + \beta \\
  x \sin \alpha + y \cos \alpha + \gamma \\
  \theta + \alpha
\end{bmatrix},
$$

where $\alpha, \beta, \gamma \in \mathbb{R}$ are arbitrary constants.