CALIFORNIA INSTITUTE OF TECHNOLOGY
Control and Dynamical Systems

ACM/CDS 202
Problem Set #5

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Reading: Abraham, Marsden, and Ratiu (MTA), section 4.2, 4.4

Problems:

1. Consider the following vector fields on $\mathbb{R}^3$:

$$X(x) = \frac{\partial}{\partial x_2} - x_1 \frac{\partial}{\partial x_3} \qquad Y(x) = \frac{\partial}{\partial x_1}.$$

Let $x_0 = (0,0,0)$. Show that $\phi_h^{-Y} \circ \phi_h^{-X} \circ \phi_h^{Y} \circ \phi_h^{X}(x_0) = h^2 \phi^{[X,Y]}(x_0)$.

2. Show that if Δ is a distribution of the form

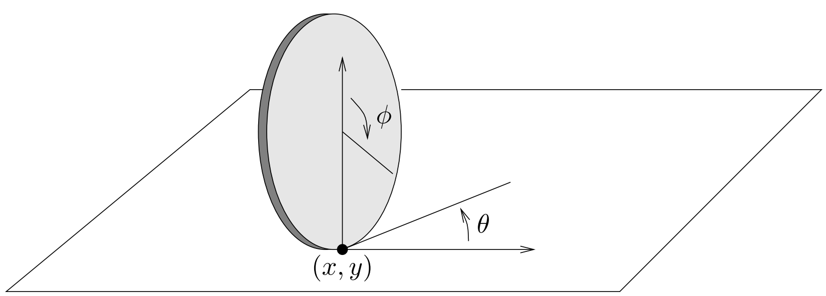
$$\Delta = \operatorname{span}\{X_1, \dots, X_d\}$$

and we have $[X_i, X_j] \in \Delta$ for all i, j then for *any* $X, Y \in \Delta$, $[X, Y] \in \Delta$. That is, to check involutivity of a distribution, we need only check that the pairwise brackets between basis elements lie in the distribution.

3. [Boothby, page 164, #4] Let $N \subset M$ be a submanifold and let $X, Y \in \mathcal{X}(M)$ be vector fields such that $X_p, Y_p \in T_pN$ for $p \in N$. Show that $[X, Y]_p \in T_pN$ for all $p \in N$.

4. [Boothby, page 164, #5] Let $F : M \to N$ be a smooth submersion of M onto N. Show that $F^{-1}(q)$ for all $q \in N$ are the leaves of a foliation on M.

5. Consider the motion of a disk rolling on the plane, as shown below:



We can represent the configuration of the disk by the xy location of the disk and the angle of the disk with respect to a fixed line on the plane. We ignore the angle through which the disk rolls. We model the motion of the disk using a one vector field to represent the *drive* input (rolling) and another vector field to represent the *steer* input (twisting).

Let $M = \mathbb{R}^2 \times S^1$ and let $p = (x, y, \theta)$ represent a point on M. Consider the two vector fields

$$X(p) = \cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y} \qquad Y(p) = \frac{\partial}{\partial \theta}$$

(X is the drive vector field and Y is the steer vector field.)

(a) Compute the flows of X and Y. Are they complete?

(b) Verify that X and Y are invariant under their own flows.

(c) Compute the Lie bracket between X and Y. Show that the tangent vectors X_p, Y_p, and $[X,Y]_p$ span T_pM for all $p \in M$.

(d) Consider the change of coordinates $z = \phi(x)$ given by

$$\begin{bmatrix} z_1 & z_2 & z_3 \end{bmatrix} = \begin{bmatrix} x\cos\theta + y\sin\theta \\ x\sin\theta - y\cos\theta \\ \theta \end{bmatrix}.$$

(The new coordinates have the physical interpretation of being the origin of the spatial reference frame when viewed from a coordinate frame attached to the disk).

Using the pushforward map for $\phi : \mathbb{R}^3 \to \mathbb{R}^3$, compute the X and Y vector fields in this new set of coordinates. (Hint: the final answer has a pretty simple form.)

(e) Show that $\phi_*[X,Y] = [\phi_* X, \phi_* Y]$ by explicit calculation.

(f) Show that X and Y are invariant under the group of rigid motions on $\mathbb{R}^2$, given by mappings

$$F(x,y,\theta) = \begin{bmatrix} x\cos\alpha - y\sin\alpha + \beta \\ x\sin\alpha + y\cos\alpha + \gamma \\ \theta + \alpha \end{bmatrix},$$

where $\alpha, \beta, \gamma \in \mathbb{R}$ are arbitrary constants.