1. If \( F(x_1, x_2, x_3) = 0 \) is a submersion defining a 2-dimensional manifold in \( \mathbb{R}^3 \), under what conditions is
\[
X = v_1 \frac{\partial}{\partial x_1} + v_2 \frac{\partial}{\partial x_2} + v_3 \frac{\partial}{\partial x_3}
\]
(evaluated at a point where \( F(x_1, x_2, x_3) = 0 \)) a tangent vector to \( M \)?

2. MTA 4.1-5: convolution equation

3. [Boothby, page 126, #6]
   Show that \( \phi_t(x, y) \) defined by
   \[
   \phi_t(x, y) = (xe^{2t}, ye^{-3t})
   \]
defines a \( C^\infty \) flow on \( M = \mathbb{R}^2 \). Determine the vector field that generates this flow (called the infinitesimal generator of the flow) and show that it is \( \phi \) invariant.

4. Let \( SO(3) \) be the set of \( 3 \times 3 \) orthogonal matrices with determinant +1. The tangent space of \( SO(3) \) at the identity is given by the set of skew-symmetric matrices of the form
   \[
   \hat{\omega} = (\omega)^\wedge = \begin{bmatrix}
   0 & -\omega_3 & \omega_2 \\
   \omega_3 & 0 & -\omega_1 \\
   -\omega_2 & \omega_1 & 0
   \end{bmatrix}
   \]
   (we’ll show this later in the course).
   (a) Show that if \( v \in \mathbb{R}^3 \), \( \hat{\omega}v = \omega \times v \), where \( \times \) is the cross product in \( \mathbb{R}^3 \).
   (b) Show that the tangent space \( T_R SO(3) \) consists of matrices of the form \( \hat{\omega}R \) where \( \hat{\omega} \) is skew-symmetric.
   (c) Show that the flow of a vector field \( g(R) = \hat{\omega}R \) is given by \( \phi_t(R) = \exp(\hat{\omega}t)R \) where \( \exp \) is the matrix exponential.

5. In this problem we will explore the properties of the Lie bracket between two vector fields, \( [X, Y] = XY - YX \).
   (a) Let \( X, Y \) be two smooth vector fields on \( M \). Show that \( XY : C^\infty(M) \to C^\infty(M) \) is not a vector field but that \( XY - YX \) is.
   (b) Let \( X, Y \in \mathcal{X}(M) \) and \( f, g \in C^\infty(M) \). Prove that
   \[
   [fX, gY] = fg[X, Y] + f(Xg)Y - g(Yf)X.
   \]