CALIFORNIA INSTITUTE OF TECHNOLOGY
Computing and Mathematical Sciences

CDS 131

Homework Set #4

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Note: In the upper left hand corner of the *second* page of your homework set, please put the number of hours that you spent on this homework set (including reading).

1. [OBC 2.1]

   (a) Let $G_1, G_2, \ldots, G_k$ be a set of row vectors on a $\mathbb{R}^n$. Let F be another row vector on $\mathbb{R}^n$ such that for every $x \in \mathbb{R}^n$ satisfying $G_i x = 0$, $i = 1, \ldots, k$, we have $Fx = 0$. Show that there are constants $\lambda_1, \lambda_2, \ldots, \lambda_k$ such that

   $$F = \sum_{i=1}^{k} \lambda_i G_i.$$

   (b) Let $x^* \in \mathbb{R}^n$ be an the extremal point (maximum or minimum) of a function f subject to the constraints $g_i(x) = 0$, $i = 1, \ldots, k$. Assuming that the gradients $\partial g_i(x^*)/\partial x$ are linearly independent, show that there are k scalers λ_i, $i = 1, \ldots, k$ such that

   $$\tilde{f}(x^*) = f(x^*) + \sum_{i=1}^{k} \lambda_i g_i(x^*).$$

2. [OBC 2.9] Consider the optimal control problem for the system

   $$\dot{x} = ax + bu \qquad J = \tfrac{1}{2} \int_{t_0}^{t_f} u^2(t)\, dt + \tfrac{1}{2} c x^2(t_f),$$

   where $x \in \mathbb{R}$ is a scalar state, $u \in \mathbb{R}$ is the input, the initial state $x(t_0)$ is given, and $a, b \in \mathbb{R}$ are positive constants. We take the terminal time t_f as given and let $c > 0$ be a constant that balances the final value of the state with the input required to get to that position. The optimal trajectory is derived in Example 4.2.

   Now consider the infinite horizon cost

   $$J = \tfrac{1}{2} \int_{t_0}^{\infty} u^2(t)\, dt$$

   with $x(t)$ at $t = \infty$ constrained to be zero.

   (a) Solve for $u^*(t) = -bPx^*(t)$ where P is the positive solution corresponding to the algebraic Riccati equation. Note that this gives an explicit feedback law ($u = -bPx$).

   (b) Plot the state solution of the finite time optimal controller for the following parameter values

   $$a = 2, \qquad b = 0.5, \qquad x(t_0) = 4,$$
   $$c = 0.1,\ 10, \qquad t_f = 0.5,\ 1,\ 10.$$

   (This should give you a total of 6 curves.) Compare these to the infinite time optimal control solution. Which finite time solution is closest to the infinite time solution? Why?

3. Consider the normalized, linearized inverted pendulum which is described by

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u = Ax + Bu.$$

Find a state feedback that minimizes the quadratic cost function

$$J = \int_0^\infty \left(q_1 x_1^2 + q_2 x_2^2 + q_u u^2\right) dt$$

where $q_1 \geq 0$ is the penalty on position, $q_2 \geq 0$ is the penalty on velocity, and $q_u > 0$ is the penalty on control actions. Show that the closed loop characteristic polynomial has the form $s^2 + 2\zeta_0\omega_0\, s + \omega_0^2$ and that $\omega_0 \geq 1$ and $\zeta_0 \geq 2/\sqrt{2}$.

4. Consider the Riccati equation

$$-\frac{dS}{dt} = A^T S + SA - SBQ_u^{-1}B^T S + Q_x, \qquad S(t_f) = Q_\mathrm{f},$$

which is quadratic in S. Show that the solution is

$$S(t) = [\Psi_{21}(t) + \Psi_{22}(t)Q_\mathrm{f}][\Psi_{11}(t) + \Psi_{12}(t)Q_\mathrm{f}]^{-1},$$

where the matrix Ψ satisfies the (*linear*) differential equation

$$\frac{d\Psi}{dt} = \frac{d}{dt}\begin{pmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{21} & \Psi_{22} \end{pmatrix} = \begin{pmatrix} A & -BQ_u^{-1}B^T \\ -Q_x & -A^T \end{pmatrix}\begin{pmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{21} & \Psi_{22} \end{pmatrix},$$

with final conditions

$$\Psi(t_f) = \begin{pmatrix} \Psi_{11}(t_f) & \Psi_{12}(t_f) \\ \Psi_{21}(t_f) & \Psi_{22}(t_f) \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$