1. Consider a closed loop system with process dynamics and a PI controller modeled by
\[ \frac{dy}{dt} + ay = bu, \quad u = k_p(r - y) + k_i \int_0^t (r(\tau) - y(\tau)) d\tau, \]
where \( r \) is the reference, \( u \) is the control variable, and \( y \) is the process output.

(a) Derive a differential equation relating the output \( y \) to the reference \( r \) by direct manipulation of the equations and compute the transfer function \( H_{yr}(s) \). Make the derivations both by direct manipulation of the differential equations and by polynomial algebra.

(b) Draw a block diagram of the system and derive the transfer functions of the process \( P(s) \) and the controller \( C(s) \).

(c) Use block diagram algebra to compute the transfer function from reference \( r \) to output \( y \) of the closed loop system and verify that your answer matches your answer in part (a).

2. Show that the transfer function of a system depends only on the dynamics in the reachable and observable subspace of the Kalman decomposition. (Hint: Consider the representation given by equation (8.20).)

3. Consider the linear state space system
\[ \frac{dx}{dt} = Ax + Bu, \quad y = Cx. \]

(a) Show that the transfer function is
\[ G(s) = \frac{b_1s^{n-1} + b_2s^{n-2} + \cdots + b_n}{s^n + a_1s^{n-1} + \cdots + a_n}, \]
where the coefficients for the numerator polynomial are linear combinations of the Markov parameters \( CA^iB \), \( i = 0, \ldots, n-1 \):
\[ b_1 = CB, \quad b_2 = CAB + a_1CB, \quad \ldots, \quad b_n = C A^{n-1}B + a_1 C A^{n-2}B + \cdots + a_{n-1}CB \]
and \( \lambda(s) = s^n + a_1s^{n-1} + \cdots + a_n \) is the characteristic polynomial for \( A \).

(b) Compute the transfer function for a linear system in reachable canonical form and show that it matches the transfer function given above.
4. Consider a closed loop system of the form of Figure 9.6, with $F = 1$ and $P$ and $C$ having a pole/zero cancellation. Show that if each system is written in state space form, the resulting closed loop system is not reachable and not observable.

5. [Dullerud and Paganini, 2.19] Consider a transfer function $G(s)$ and let $(A, B, C, D)$ be a realization: 

$$G(s) = C(sI - A)^{-1}B + D.$$ 

(a) Prove that if $(A, B, C, D)$ is a minimal realization for $G(s)$ then every eigenvalue of $A$ must be a pole of $G(s)$.

(b) Prove that if $(A, B, C, D)$ is a non-minimal realization, then every eigenvalue of $A$ is either a pole of $G(s)$ or corresponds to an eigenvector that is in the unreachable or unobservable subspace.

6. Complete the proof of Theorem 1 in DFT by showing that if the polynomial $NPNC + MPMC$ has a zero $s_0$ in the right half plane, then it is not possible for $s_0$ to also a zero of all nine numerators in equation (3.2). Hence if $NPNC + MPMC$ has an unstable zero (pole of the system), at least one of the 4 transfer functions used to check internal stability for the pair $(P, C)$ will be unstable.