1. Consider the double integrator system \( \dddot{y} = u \). Use the controllability Gramian to compute an input that steers the system for the origin to a state \( x_f \) in time \( T \). What happens as \( T \to 0 \) and as \( T \to \infty \)?

2. [FBS 7.2] Extend the argument in Section 7.1 in Feedback Systems to show that if a system is reachable from an initial state of zero, it is reachable from a nonzero initial state.

3. Consider a system with the state \( x \) and \( z \) described by the equations

\[
\frac{dx}{dt} = Ax + Bu, \quad \frac{dz}{dt} = Az + Bu.
\]

If \( x(0) = z(0) \) it follows that \( x(t) = z(t) \) for all \( t \) regardless of the input that is applied. Assuming that the pair \((A, B)\) is controllable, compute the rank of the reachability Grammian \( W_c \) and use this to determine the reachable space of the system starting from the origin and its dimension.

4. [FBS 7.10] Let \( A \in \mathbb{R}^{n \times n} \) be a matrix with characteristic polynomial \( \lambda(s) = \det(sI - A) = s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n \). Show that the matrix \( A \) satisfies

\[
\lambda(A) = A^n + a_1 A^{n-1} + \cdots + a_{n-1} A + a_n I = 0,
\]

where the zero on the right hand side represents a matrix of elements with all zeros. Use this result to show that \( A^n \) can be written in terms of lower order powers of \( A \) and hence any matrix polynomial in \( A \) can be rewritten using terms of order at most \( n - 1 \).

5. [Sontag 3.3.4] Assume that the pair \((A, B)\) is not controllable with \( \dim R(A, B) = \text{rank} W_c = r < n \). From Lemma 3.3.3, there exists an invertible matrix \( T \in \mathbb{R}^{n \times n} \) such that the matrices \( \tilde{A} := T^{-1} A T \) and \( \tilde{B} := T^{-1} B \) have the block structure

\[
\tilde{A} = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} B_1 \\ 0 \end{pmatrix},
\]

where \( A_1 \in \mathbb{R}^{r \times r} \) and \( B_1 \in \mathbb{R}^{r \times m} \). Prove that \((A_1, B_1)\) is itself a controllable pair.

6. [Sontag 3.3.6] Prove that if

\[
A = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & \lambda_n \end{pmatrix}, \quad B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-1} \\ b_n \end{pmatrix}
\]

then \((A, B)\) is controllable if and only if \( \lambda_i \neq \lambda_j \) for each \( i \neq j \) and all \( b_i \neq 0 \).