CALIFORNIA INSTITUTE OF TECHNOLOGY
Computing and Mathematical Sciences

CDS 131

R. Murray | **Homework Set #3** | Issued: 16 Oct 2019
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Note: In the upper left hand corner of the *second* page of your homework set, please put the number of hours that you spent on this homework set (including reading).

1. Consider the double integrator system $\ddot{y} = u$. Use the controllability Gramian to compute an input that steers the system for the origin to a state x_f in time T. What happens as $T \to 0$ and as $T \to \infty$?

2. [FBS 7.2] Extend the argument in Section 7.1 in *Feedback Systems* to show that if a system is reachable from an initial state of zero, it is reachable from a nonzero initial state.

3. Consider a system with the state x and z described by the equations

$$\frac{dx}{dt} = Ax + Bu, \qquad \frac{dz}{dt} = Az + Bu.$$

If $x(0) = z(0)$ it follows that $x(t) = z(t)$ for all t regardless of the input that is applied. Assuming that the pair (A, B) is controllable, compute the rank of the reachability Grammian W_c and use this to determine the reachable space of the system starting from the origin and its dimension.

4. [FBS 7.10] Let $A \in \mathbb{R}^{n \times n}$ be a matrix with characteristic polynomial $\lambda(s) = \det(sI - A) = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n$. Show that the matrix A satisfies

$$\lambda(A) = A^n + a_1 A^{n-1} + \cdots + a_{n-1}A + a_n I = 0,$$

where the zero on the right hand side represents a matrix of elements with all zeros. Use this result to show that A^n can be written in terms of lower order powers of A and hence any matrix polynomial in A can be rewritten using terms of order at most $n - 1$.

5. [Sontag 3.3.4] Assume that the pair (A, B) is not controllable with $\dim R(A, B) = \operatorname{rank} W_c = r < n$. From Lemma 3.3.3, there exists an invertible matrix $T \in \mathbb{R}^{n \times n}$ such that the matrices $\tilde{A} := T^{-1}AT$ and $\tilde{B} := T^{-1}B$ have the block structure

$$\tilde{A} = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}, \qquad \tilde{B} = \begin{pmatrix} B_1 \\ 0 \end{pmatrix},$$

where $A_1 \in \mathbb{R}^{r \times r}$ and $B_1 \in \mathbb{R}^{r \times m}$. Prove that (A_1, B_1) is itself a controllable pair.

6. [Sontag 3.3.6] Prove that if

$$A = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & \lambda_n \end{pmatrix}, \qquad B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-1} \\ b_n \end{pmatrix}$$

then (A, B) is controllable if and only if $\lambda_i \neq \lambda_j$ for each $i \neq j$ and all $b_i \neq 0$.