1. [FBS 6.1] Show that if \( y(t) \) is the output of a linear system corresponding to input \( u(t) \), then the output corresponding to an input \( \dot{u}(t) \) is given by \( \dot{y}(t) \). (Hint: Use the definition of the derivative: \( \dot{z}(t) = \lim_{\epsilon \to 0} (z(t+\epsilon) - z(t))/\epsilon \).)

2. [FBS 6.2] Show that a signal \( u(t) \) can be decomposed in terms of the impulse function \( \delta(t) \) as

\[
u(t) = \int_0^t \delta(t-\tau)u(\tau)\,d\tau
\]

and use this decomposition plus the principle of superposition to show that the response of a linear, time-invariant system to an input \( u(t) \) (assuming a zero initial condition) can be written as

\[
y(t) = \int_0^t h(t-\tau)u(\tau)\,d\tau,
\]

where \( h(t) \) is the impulse response of the system. (Hint: Use the definition of the Riemann integral.)

3. [FBS 6.4] Assume that \( \zeta < 1 \) and let \( \omega_\text{d} = \omega_0 \sqrt{1-\zeta^2} \). Show that

\[
\exp\begin{pmatrix}
-\zeta \omega_0 & \omega_\text{d} \\
-\omega_\text{d} & -\zeta \omega_0
\end{pmatrix} t = e^{-\zeta \omega_0 t} \begin{pmatrix}
\cos \omega_\text{d} t & \sin \omega_\text{d} t \\
-\sin \omega_\text{d} t & \cos \omega_\text{d} t
\end{pmatrix}.
\]

Also show that

\[
\exp\begin{pmatrix}
-\omega_0 & \omega_0 \\
0 & -\omega_0
\end{pmatrix} t = e^{-\omega_0 t} \begin{pmatrix}
1 & \omega_0 t \\
0 & 1
\end{pmatrix}.
\]

4. [FBS 6.6] Consider a linear system with a Jordan form that is non-diagonal.

(a) Prove Proposition 6.3 by showing that if the system contains a real eigenvalue \( \lambda = 0 \) with a nontrivial Jordan block, then there exists an initial condition with a solution that grows in time.

(b) Extend this argument to the case of complex eigenvalues with \( \text{Re} \lambda = 0 \) by using the block Jordan form

\[
J_i = \begin{pmatrix}
0 & \omega & 1 & 0 \\
-\omega & 0 & 0 & 1 \\
0 & 0 & 0 & \omega \\
0 & 0 & -\omega & 0
\end{pmatrix}.
\]
5. **[FBS 6.8]** Consider a linear discrete-time system of the form

\[ x[k + 1] = Ax[k] + Bu[k], \quad y[k] = Cx[k] + Du[k]. \]

(a) Show that the general form of the output of a discrete-time linear system is given by the discrete-time convolution equation:

\[ y[k] = CA^k x[0] + \sum_{j=0}^{k-1} CA^{k-j-1} Bu[j] + Du[k]. \]

(b) Show that a discrete-time linear system is asymptotically stable if and only if all the eigenvalues of \( A \) have a magnitude strictly less than 1.

6. Consider a stable linear time-invariant system. Assume that the system is initially at rest and let the input be \( u = \sin \omega t \), where \( \omega \) is much larger than the magnitudes of the eigenvalues of the dynamics matrix. Show that the output is approximately given by

\[ y(t) \approx |G(i\omega)| \sin (\omega t + \arg G(i\omega)) + \frac{1}{\omega} h(t), \]

where \( G(s) \) is the frequency response of the system and \( h(t) \) its impulse response.

7. Consider the system

\[ \frac{dx}{dt} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u, \quad y = \begin{pmatrix} 1 & 0 \end{pmatrix} x, \]

which is stable but not asymptotically stable. Show that if the system is driven by the bounded input \( u = \cos t \) then the output is unbounded.