CALIFORNIA INSTITUTE OF TECHNOLOGY
Control and Dynamical Systems

CDS 110b

R. M. Murray | **Problem Set #5** | Issued: 13 Feb 08
Winter 2008 | | Due: 20 Feb 08

Note: Please put the number of hours that you spent on this homework set (including reading) on the back of the first page of your homework.

1. Consider a control system having state space dynamics

$$\frac{dx}{dt} = \begin{bmatrix} -\alpha-\beta & 1 \\ -\alpha\beta & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ k \end{bmatrix} u, \qquad y = \begin{bmatrix} 1 & 0 \end{bmatrix} x.$$

   (a) Construct an observer for the system and find expressions for the observer gain $L = \begin{bmatrix} l_1 & l_2 \end{bmatrix}^T$ such that the observer has natural frequency ω_0 and damping ratio ζ.

   (b) Suppose that we choose a different output

$$\tilde{y} = \begin{bmatrix} 1 & \gamma \end{bmatrix} x.$$

   Are there any values of γ for which the system is *not* observable? If so, provide an example of an initial condition and output where it is not possible to uniquely determine the state of the system by observing its inputs and outputs.

2. (OBC, 4.3) Consider a second order system with dynamics

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -a & 0 \\ 0 & -b \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} v, \qquad y = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

   that is forced by Gaussian white noise with zero mean and variance σ^2. Assume $a, b > 0$.

   (a) Compute the correlation function $\rho(\tau)$ for the output of the system. Your answer should be an explicit formula in terms of a, b and σ.

   (b) Assuming that the input transients have died out, compute the mean and variance of the output.

Students who are not doing the course project should complete the following additional problems:

3. (OBC, 4.1) A random variable Y is the sum of two independent normally (Gaussian) distributed random variables having means m_1, m_2 and variances σ_1^2, σ_2^2 respectively. Show that the probability density function for Y is

$$p(y) = \frac{1}{2\pi\sigma_1\sigma_2} \int_{-\infty}^{\infty} \exp\left\{ -\frac{(y-x-m_1)^2}{2\sigma_1^2} - \frac{(x-m_2)^2}{2\sigma_2^2} \right\} dx$$

   and confirm that this is normal (Gaussian) with mean $m_1 + m_2$ and variance $\sigma_1^2 + \sigma_2^2$. (Hint: Use the fact that $p(z|y) = p_x(x) = p_x(z-y)$.)

4. (OBC, 4.2) Consider the motion of a particle that is undergoing a random walk in one dimension (i.e., along a line). We model the position of the particle as

$$x[k+1] = x[k] + u[k],$$

where x is the position of the particle and u is a white noise processes with $E\{u[i]\} = 0$ and $E\{u[i]\, u[j]\} R_u \delta(i-j)$. We assume that we can measure x subject to additive, zero-mean, Gaussian white noise with covariance 1. Show that the expected value of the particle as a function of k is given by

$$E\{x[k]\} = E\{x[0]\} + \sum_{i=0}^{k-1} E\{u[i]\} = E\{x[0]\} =: \mu_x$$

and the coveriance $E\{(x[k] - \mu_x)^2\}$ is given by

$$E\{(x[k] - \mu_x)^2\} = \sum_{i=0}^{k-1} E\{u^2[i]\} = kR_u$$