CALIFORNIA INSTITUTE OF TECHNOLOGY
Control and Dynamical Systems

CDS 110b

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Problem Set #3

Issued: 23 Jan 08
Due: 30 Jan 08

Note: Please put the number of hours that you spent on this homework set (including reading) on the back of the first page of your homework.

1. (OBC, 2.8) Consider the control system transfer function

$$H(s) = \frac{s+b}{s(s+a)} \qquad a, b > 0$$

with state space representation

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & -a \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} b & 1 \end{bmatrix} x$$

and performance criterion

$$V = \int_0^\infty (x_1^2 + u^2) dt.$$

(a) Let

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$$

with $p_{12} = p_{21}$ and $P > 0$ (positive definite). Write the steady state Riccati equation as a system of four explicit equations in terms of the elements of P and the constants a and b.

(b) Find the gains for the optimal controller assuming the full state is available for feedback.

(c) Find the closed loop natural frequency and damping ratio.

2. (OBC, 2.11) Consider the lateral control problem for an autonomous ground vehicle from Example 1.1. We assume that we are given a reference trajectory $r = (x_d, y_d)$ corresponding to the desired trajectory of the vehicle. For simplicity, we will assume that we wish to follow a straight line in the x direction at a constant velocity $v_d > 0$ and hence we focus on the y and θ dynamics:

$$\dot{y} = \sin\theta \, v_d$$
$$\dot{\theta} = \frac{1}{\ell} \tan\phi \, v_d.$$

We let $v_d = 10$ m/s and $\ell = 2$ m.

(a) Design an LQR controller that stabilizes the position y to the origin. Plot the step and frequency response for your controller and determine the overshoot, rise time, bandwidth and phase margin for your design. (Hint: for the frequency domain specifications, break the loop just before the process dynamics and use the resulting SISO loop transfer function.)

(b) Suppose now that $y_d(t)$ is not identically zero, but is instead given by $y_d(t) = r(t)$. Modify your control law so that you track $r(t)$ and demonstrate the performance of your controller on a "slalom course" given by a sinusoidal trajectory with magnitude 1 meter and frequency 1 Hz.

Students who are not doing the course project should complete the following additional problems:

3. (Friedland 9.6) Consider the dynamics of a DC motor driving an inertial load (see ÅM08, Exercise 2.10 for a picture of a similar system):

$$\begin{aligned} \dot{\theta} &= \omega \\ \dot{\omega} &= -\alpha\omega + \beta u \end{aligned}$$

where θ is the angular position of the load, ω is the angular velocity, u is the applied voltage, and α and β are constants that depend on the physical parameters of the motor and load. For this problem, let $\alpha = 1$ and $\beta = 3$.

(a) Let $e = \theta - \theta_d$. For the performance criterion

$$V = \int_0^\infty (q_1^2 e^2 + u^2)\, d\tau$$

find and tabulate the control gains and corresponding closed-loop poles for $q_1 = 0.1, 1, 10$.

(b) Plot the transient response (e as a function of t) for the initial error of unity for the values of q_1 in part (a). (Note: you should use the MATLAB `initial` function to get the transient response to an initial error. Make sure to set the initial condition appropriately.)

(c) In addition to weighting the position error it is also desired to limit the velocity by using a performance criterion

$$V = \int_0^\infty (q_1^2 e^2 + q_2^2 \dot{e}^2 + u^2)\, d\tau.$$

For the values of q_1^2 used in part (a) and $q_2^2 = 0.1q_1^2, q_1^2, 10q_1^2$ find the control gains and corresponding closed loop poles.

(d) Plot the transient response as in part (b) for a range of q_1^2 and q_2^2 (you need not include all 9 plots; just the "interesting" ones). Compare the results with those of part (b). Are the results as expected?

4. (Friedland 9.10) Consider the motor-driven inverted pedulum on a cart, whose linearized dynamics are given by

$$\begin{aligned} \ddot{x} + \frac{k^2}{Mr^2R}\dot{x} + \frac{mg}{M}\theta &= \frac{k}{MRr}u \\ \ddot{\theta} - \left(\frac{M+m}{Ml}\right) g\theta - \frac{k^2}{Mr^2Rl}\dot{x} &= -\frac{k}{MRrl}u \end{aligned}$$

where k is the motor torque constant, R is the motor resistance, r is the ratio of the linear forces applied to the cart ($\tau = rf$), and u is the voltage applied to the motor. The following numerical data may be used:

$$m = 0.1 \text{ kg} \qquad M = 1.0 \text{ kg} \qquad l = 1.0 \text{ m} \qquad g = 9.8 \text{ m/s}^2$$
$$k = 1 \text{ V} \cdot \text{s} \qquad R = 100\ \Omega \qquad r = 0.02 \text{ m}$$

We wish to optimize the gains using a performance criterion of the form

$$V = \int_0^\infty \left(q_1^2 x^2 + q_3^2 \theta^2 + \rho^2 u^2\right)\, dt$$

A pendulum angle much greater than 1 degree = 0.017 rad would be precarious. Thus a heavy weighting error on θ is indicated: $q_3^2 = 1/(0.017)^2 \approx 3000$. For the physical dimensions of the system, a position error of the order of 10 cm = 0.1 m is not unreasonable. Hence $q_1^2 = 1/(0.1)^2 = 100$.

(a) Using these values of q_1^2 and q_3^2, determine and plot the gain matrices and corresponding closed loop poles as a function of the control weighting parameter ρ^2 for $0.001 < \rho^2 < 50$.

(b) Repeat part (a) for a heavier weighting: $q_1^2 = 10^4$ on the cart displacement.

(c) Plot the step responses for the controllers you defined in parts (a) and (b) and explain their behavior in terms of the cost functions you used.