Suppose we have performance specs on $S, T, CS, PS$ (gang of 4) when do we satisfy these specs for $\tilde{P} = P + \Delta$ (or other types of uncertainty)?

**Disturbance attenuation:** $G_{yd}$

$$G_{yd} = \frac{P}{1 + PC} = PS$$

$$G_{yd}(P + \Delta) \approx G_{yd}(P) + \frac{dG_{yd}}{dP} \Delta$$

$$\frac{dG_{yd}}{dP} = \frac{1}{1 + PC} - \frac{P}{(1 + PC)^2} C = \frac{1 + PC - PC}{(1 + PC)^2} = \frac{1}{(1 + PC)^2} = \frac{G_{yd}}{P} S$$

$$\frac{dG_{yd}}{G_{yd}} = S \frac{dP}{P} \leftarrow S \text{ determines "sensitivity" to perturbations}$$

**Noise attenuation:** $\frac{dG_{un}}{G_{un}} = T \frac{dP}{P}$

**Remarks**

1. As in CDS 110a, would like to keep both $T$ & $S$ small, but $S + T = 1$

2. Can also compute $\frac{dG_{yr}}{dCyr} = S \frac{dP}{P}$ and other gang of 6 sensitivities
Performance Specification via Weighted Sensitivity

\[ H_\omega = \frac{1}{1 + PC} = S \]

Use frequency weighting to specify performance over given range
- Doesn't make sense to ask for small error over all frequencies
- Plant can't track reference at extremely high frequencies

\[ |S(j\omega)| \leq \frac{1}{|W_1(j\omega)|} \]

\[ |W_1(j\omega)S(j\omega)| \leq 1 \]

Example: Ducted fan

Parameter values
- \( J = 0.0475 \) inertia around pitch axis, kg m\(^2\)
- \( m = 1.5 \) mass of fan, kg
- \( r = 0.25 \) distance to flaps, m
- \( g = 10 \) gravitational constant, m/s\(^2\)
- \( d = 0.1 \) damping factor (estimated)
- \( l = 0.05 \) offset of center of mass, m

\[ C(s) = 20 \frac{(s + 25)}{(s + 300)} \]

\[ W_1 = \frac{20}{(s/12 + 1)^2} \]

Bode Plot

Phase Angle vs Magnitude (dB)
Preview: Robust Performance

\[ |S(j\omega)| \leq \frac{1}{|W_1(j\omega)|} \]

**Thm** C provides robust performance to multiplicative uncertainty if and only if

\[ \max_{\omega}(|W_1^* S| + |W_2 T|) < 1 \]

**Remarks**
- Gives conditions for guaranteed robust performance
- Given \( W_1 \) and \( W_2 \), still need to find C that works

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Nyquist Interpretation of Robust Performance

**Stability and performance conditions:**

1. Nyquist plot outside \(-1\)  \(\Rightarrow\) nominal stability

2. Nyquist plot outside \( W_1 \) ball  \(\Rightarrow\) nominal performance  (generalizes gain/phase margin)

3. Nyquist plot outside \( W_2 \) ball  \(\Rightarrow\) robust stability

4. Combined non-intersection  \(\Rightarrow\) robust performance

**Notes**
- Size of balls varies as frequency changes
- Condition is equivalent to
  \[ \max_{\omega}(|W_1^* S| + |W_2 T|) < 1 \]

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Example: Caltech Ducted Fan Design Problem (1/2)

\[
\begin{align*}
\dot{x} &= f_1 \cos \theta - f_2 \sin \theta \\
\dot{y} &= f_1 \sin \theta + f_2 \cos \theta - mg \\
\dot{\theta} &= r f_1 \\
\frac{\dot{\theta}}{u(s)} &= \frac{r}{Js^2 + ds + mg} \\
|S(j\omega)| &\leq \frac{1}{|W_1(j\omega)|} \\
W_1 &= \frac{20}{s(j/12 + 1)^2} \\
W_2 &= 0.2
\end{align*}
\]

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Example: Caltech Ducted Fan Design Problem (2/2)

System satisfies nominal performance
System satisfies robust stability
System does not have robust performance

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