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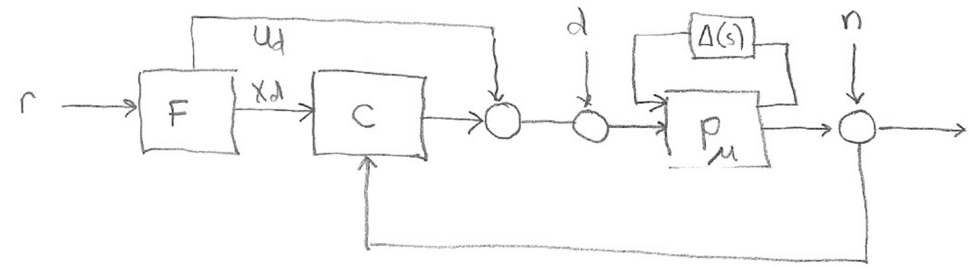
Lecture 9-1: Robust Stability

Sources of uncertainty in control systems design



Noise & disturbances: d & n — Gang of 4/6, Kalman Filters

Unknown parameters: μ

Unmodeled dynamics: $\Delta(s)$

Focus of today's lecture

Examples of unmodeled dynamics:

$$\tilde{P}(s) = \frac{\gamma}{s+a} = \frac{\gamma+\delta}{s+a} = \underbrace{\frac{\gamma}{s+a}}_{P(s)} + \underbrace{\frac{\delta}{s+a}}_{\Delta(s)} \qquad |\delta| < 0.1\gamma$$

known — unknown

This is called additive uncertainty: $\tilde{P}(s) = P(s) + \Delta(s)$

Can also model this as multiplicative uncertainty

$$\hat{P}(s) = \frac{\gamma+\delta}{s+a} = \frac{\gamma}{s+\alpha}\left(1+\frac{\delta}{\gamma}\right) = P(s)(1+\delta(s))$$

$$\delta(s) = \Delta(s)/P(s)$$

Here: $|\delta(s)| \leq 0.1$ bounds uncertainty

Goal: Given an uncertainty process $\tilde{P}(s) = P(s) + \Delta(s)$ (or variant), find a controller $C(s)$ such that

1) Gang of 4 are stable w/ $\hat{P}$ — Robust stability (today)
2) Gang of 4 maintain good performance — Robust performance (Wed)

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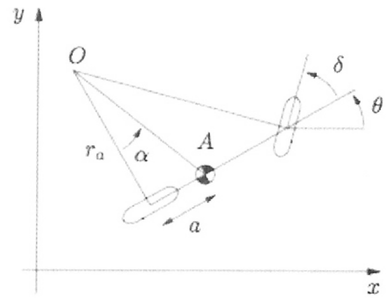
Motivating Example: Vehicle Steering (AM06)



$$\dot{\xi} = v\cos(\alpha+\theta)$$
$$\dot{\eta} = v\sin(\alpha+\theta)$$
$$\dot{\theta} = \frac{v_0}{b}\tan\delta,$$

$$\frac{dw}{d\tau} = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix} w + \begin{bmatrix}\alpha\\ 1\end{bmatrix} u$$
$$y = \begin{bmatrix}1 & 0\end{bmatrix} w$$

- Control the lateral dynamics to follow trajectory
- Normalize dynamics to simply functional form (AM06)

Control design

$$u = -k_1 w_1 - k_2 w_2 + k_r r;$$

- Choose gains so that closed loop dynamics have characteristic polynomial

$$p(s) = s^2 + 2\zeta_c\omega_c s + \omega_c^2.$$

Fast rise time $\omega_c = 1,\ \zeta_c = 0.707$

$$\Downarrow$$

$$k_1 = 100 \text{ and } k_2 = -35.86$$

Estimator design

$$\frac{d\hat{w}}{dt} = A\hat{w} + Bu + L(y - C\hat{w})$$

- Choose estimator gains to give closed loop observer with characteristic poly

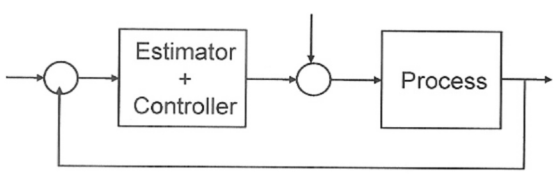
$$q(s) = s^2 + 2\zeta_o\omega_o s + \omega_o^2,$$

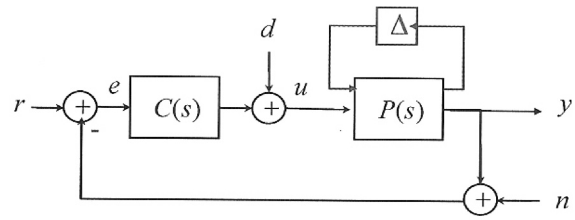
Fast $\longrightarrow$ $\omega_o = 20 \text{ and } \zeta_o = 2$ ← well damped

$$\Downarrow$$

$$l_1 = 28.28 \text{ and } l_2 = 400$$

Closed Loop Steering Dynamics

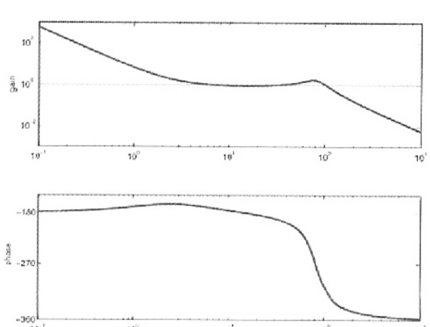




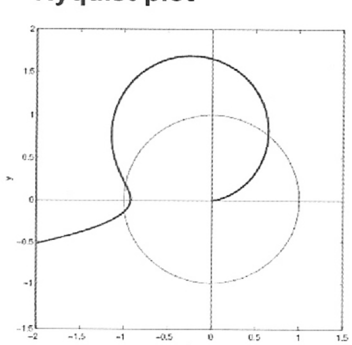
Process and controller dynamics

$$P(s) = \frac{0.5s+1}{s^2}, \qquad C(s) = \frac{-11516s + 40000}{s^2 + 42.4s + 6657.9}.$$

Bode Plot



Nyquist plot



Closed loop not robust!
- No net encirclements, but phase margin is very small (around 7 deg)
- Small change in process dynamics will cause system to go unstable
- Not clear from design what went wrong - closed loop dynamics look reasonable
- Problem is not limited to eigenvalue placement (HW: show for LQR + KF)

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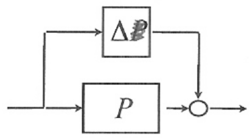
Robust Stability via Nyquist

Additive uncertainty case

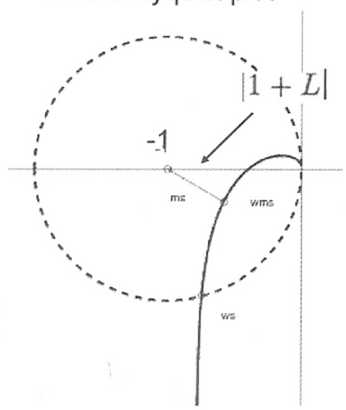


$$\tilde{P} = P + \Delta \cancel{P}$$

$$\tilde{L} = \tilde{P}C = PC + \Delta \cancel{P} C$$

How much ΔP can we tolerate?

- Look at distance from the critical point on the Nyquist plot



$$|C\Delta \cancel{P}| < |1+L|$$

$$\Downarrow$$

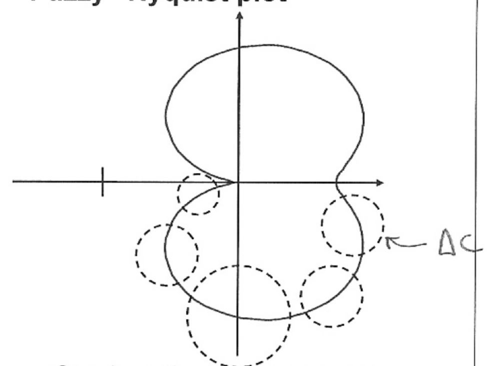
$$|\Delta \cancel{P}| < \left|\frac{1+PC}{C}\right|$$

$$\Downarrow$$

$$\left|\frac{\Delta \cancel{P}}{P}\right| < \frac{1}{|T|}$$

(must hold at all frequencies)

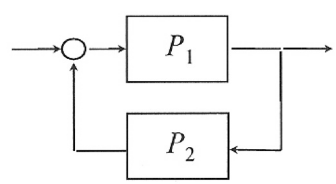
"Fuzzy" Nyquist plot



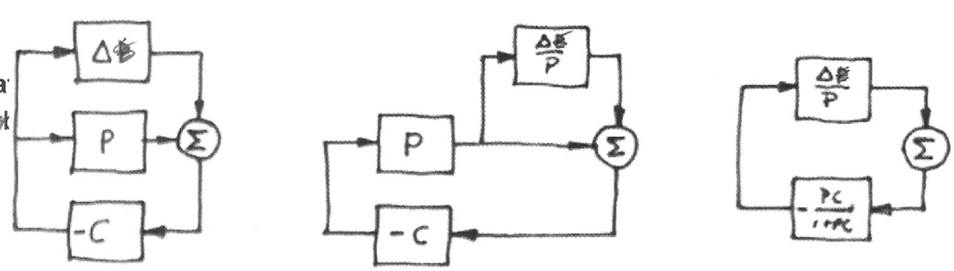
- Can handle any uncertainty within "tube" around nominal process (no new encirc'ts)
- Caution: requires perturbations to be *stable*
- Conservative condition: allows *any* variation w/in tube; reality may be more kind

Small Gain Theorem

Theorem (Small gain theorem)
Consider two
stable, linear time invariant processes with transfer
functions $P_1(s)$ and $P_2(s)$. The feedback interconnection of
these two systems is
stable if $\| P_1 P_2 \|_\infty < 1$.



~~Applica~~
- ~~Use~~



$$\left\| \frac{\Delta \cancel{P}}{P} \cdot T \right\|_\infty < 1 \implies \left|\frac{\Delta \cancel{P}}{P}\right| < \frac{1}{|T|} \quad \forall\omega$$

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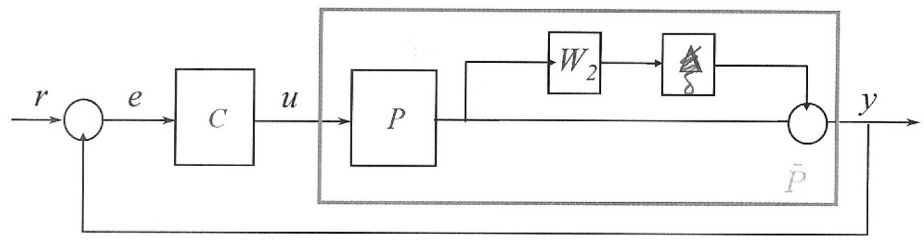
Robust Stability Using Frequency Domain Weighting



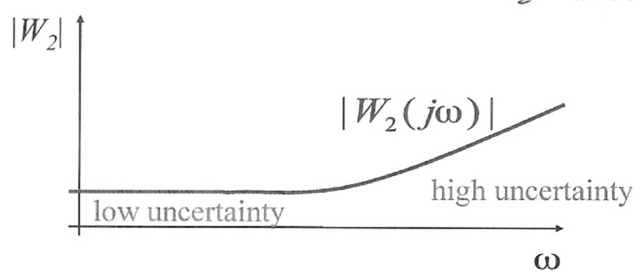
Multiplicative Uncertainty

- Model plant as nominal with additional dynamics given by $W_2 \cancel{\Delta}\, \delta$

$$\tilde{P} = P(1 + W_2 \cancel{\Delta}\,\delta)$$

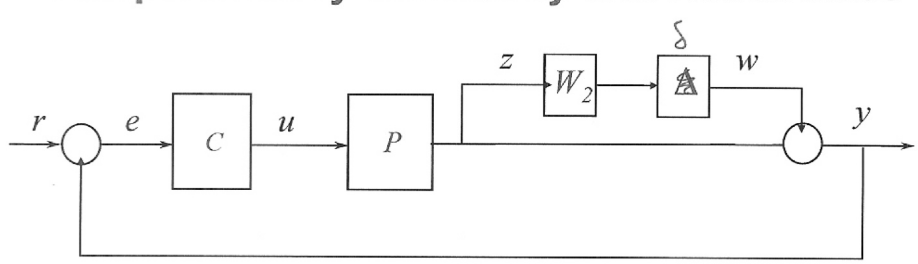
W_2 = frequency weight

$\delta\, \cancel{\Delta}$ = uncertainty; require $|\cancel{\Delta}\,\delta(jw)| \leq 1$ (normalized)



- Δ allows *any* dynamics to be inserted into the plant
- Can be used to model parameter uncertainty or unmodeled dynamics

Complementary Sensitivity and Robustness



Thm *A controller C provides robust stability to multiplicative perturbations if and only if*

$$\boxed{|W_2(j\omega)\ T(j\omega)| \cancel{<} \leq 1} \textit{ for all } \omega.$$

where Complementary sensitivity function

$$T := \frac{PC}{1+PC} = H_{zw}$$

Intuition: H_{zw} represents the transfer function seen by the weighted uncertainty $W_2\,\Delta$

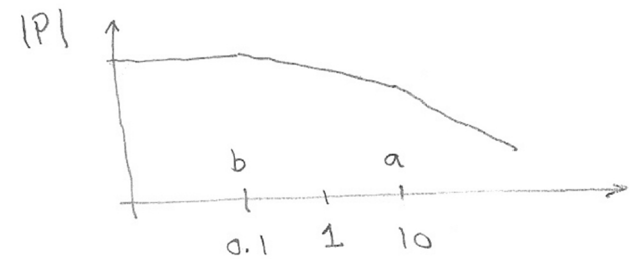
Note: this theorem guarantees stability for any *transfer function* $\Delta(s)$ satisfying $|\Delta(j\omega)| < 1 \Rightarrow$ allows unmodeled *dynamics* (as well as parametric uncertainty)

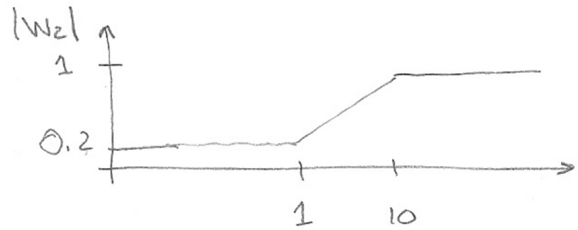
Example: cruise control w/ engine dynamics

$$P(s) = \underbrace{\frac{1/m}{s+b/m}}_{\text{car}} \cdot \underbrace{\frac{r}{s+a}}_{\text{eng}} \underbrace{(1 + W_2 \delta)}_{\text{unknown}}$$

Unknown dynamics:
- high freq engine dynamics
- unmodeled time delays

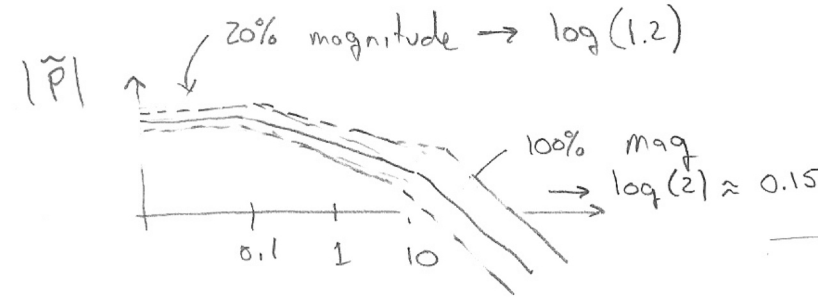
$W_2(s)$ describes unknown frequency range

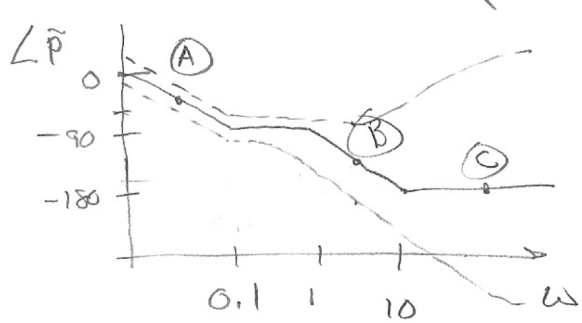


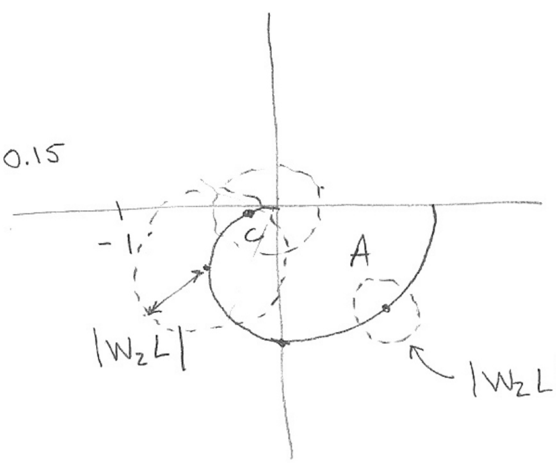


Up to 20% uncertainty at low freq

Up to 100% uncertainty at high freq







$$\tilde{L} = \tilde{P}C = P(1+W_2\delta)C$$
$$= PC + PC(W_2\delta) = L(1+W_2\delta)$$

Remarks

1. Gain & phase margin not particularly accurate measures.
2. Use stability margin: $M_s = \min_\omega |1+L| = \max \frac{1}{|1+L|} = \max_\omega |S|$

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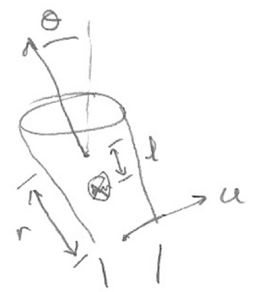
Example: ducted fan (robust stability)



$P(s) = H_{\theta u} = \dfrac{r}{Js^2 + mgl}$ $\quad r = 0.25\,m \quad g = 9.8$
$\quad J = 0.047 \quad l = 0.05$

Uncertainty #1: $r = r_0 \pm 20\%$ (dynamic)

Uncertainty #2: time delay $e^{-\tau s}$ $\quad 0 \le \tau \le 0.1$

Uncertainty #3: $l = l_0 \pm 50\%$

Robust Stability bounds: $\left|\dfrac{\Delta P}{P}\right| < \dfrac{1}{|T|}$. If $\left|\dfrac{\Delta P}{P}\right| < |W_2|$ and $|T| < \dfrac{1}{|W_2|}$ then OK

Performance bounds: $|S| < 0.05$ up to 5 Hz (30 rad/sec)

$\Rightarrow |T| > 0.95$ up to 5 Hz

$\Rightarrow \left|\dfrac{\Delta P}{P}\right| < \dfrac{1}{0.95}$ up to 5 Hz

Uncertainty # 1: $\dfrac{\Delta P}{P} = \delta \quad |\delta| < 0.2 \quad |T| < 5$

Uncertainty # 2: $\Delta P = \dfrac{r}{Js^2 + mgl}\left(e^{\tau s} - 1\right) \quad |\Delta P / P| = |e^{\tau s} - 1| \le \left|\dfrac{0.21 s}{0.1 s + 1}\right|$

Uncertainty # 3: Use feedback uncertainty (HW) $\quad |T| < \left|\dfrac{0.1 s + 1}{0.21 s}\right|$