CDS 110b: Lecture 3-1
Linear Quadratic Regulators

Richard M. Murray

23 January 2008

Goals:

- Derive the linear quadratic regulator and demonstrate its use

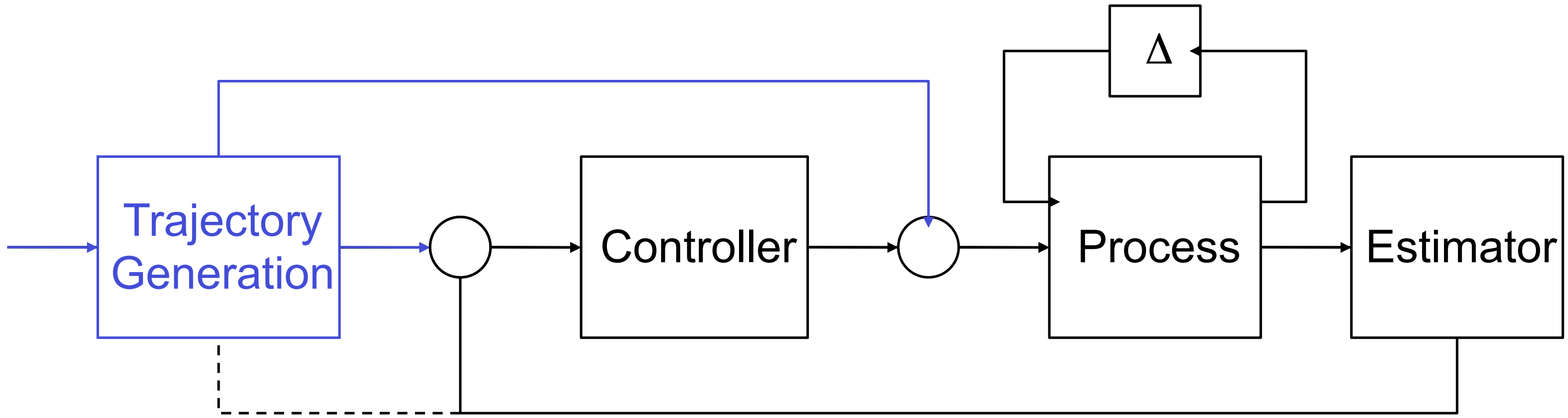
Reading:

- RMM course notes (available on web page)
- Lewis and Syrmos, Section 3.3
- Friedland, Chapter 9 (different derivation, but same result)

Homework #3

- Design LQR controllers for some representative systems
- Due Wed, 30 Jan by 5 pm, in box outside 109 Steele

Review from last lecture



Trajectory Generation via Optimal Control:

$$\dot{x} = f(x,u) \quad x = \mathbb{R}^n$$
$$x(0) \text{ given} \quad u \in \Omega \subset \mathbb{R}^p$$

$$J = \int_0^T L(x,u)\,dt + V(x(T))$$
$$\psi(x(T)) = 0$$

Today: focus on special case of a linear quadratic regulator

$$\dot{x} = Ax + Bu \quad x = \mathbb{R}^n$$
$$x(0) \text{ given} \qquad u \in \mathbb{R}^p$$

$$J = \int_0^T x^T Q x + u^T R u\,dt + x(T)^T P_1 x(T)$$

no terminal constraints

Linear Quadratic Regulator (finite time)

Problem Statement

$$\dot{x} = Ax + Bu \quad x = \mathbb{R}^n \qquad \tilde{J} = \frac{1}{2}\int_0^T \left(x^T Q x + u^T R u\right)\, dt + \frac{1}{2} x^T(T) P_1 x(T)$$

$$x(0) \text{ given} \qquad u \in \mathbb{R}^p$$

- Factor of 1/2 simplifies some math below; optimality is not affected

Solution: use the maximum principle

$$H = x^T Q x + u^T R u + \lambda^T (Ax + Bu)$$

Note: in the notes (and the lecture), we use the notation Q = Q_x and R = Q_u.

$$\dot{x} = \left(\frac{\partial H}{\partial \lambda}\right)^T = Ax + Bu \qquad x(0) = x_0$$

$$-\dot{\lambda} = \left(\frac{\partial H}{\partial x}\right)^T = Qx + A^T \lambda \qquad \lambda(T) = P_1 x(T)$$

$$0 = \frac{\partial H}{\partial u} = Ru + \lambda^T B \qquad \Longrightarrow \qquad u = -R^{-1} B^T \lambda.$$

- This is still a two point boundary value problem ⇒ hard to solve
- Note that solution is linear in x (because λ is linear in x, treated as an input)

Simplified Form of the Solution

Can simplify solution by guessing that $\lambda = P(t)\, x(t)$

$$-\dot{\lambda} = \left(\frac{\partial H}{\partial x}\right)^T = Qx + A^T\lambda \qquad \lambda(T) = P_1 x(T)$$

$\longleftarrow$ From maximum principle

$$\dot{\lambda} = \dot{P}x + P\dot{x} = \dot{P}x + P(Ax - BR^{-1}B^TP)x$$

$\longleftarrow$ Substitute $\lambda = P(t)\, x(t)$

$$\Downarrow$$

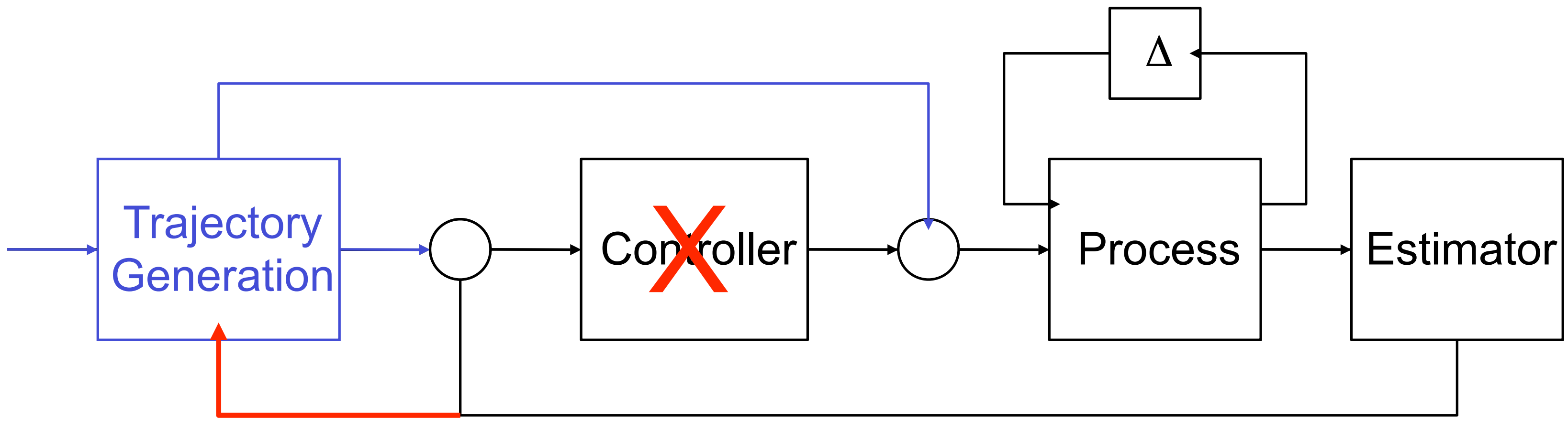
$$-\dot{P}x - PAx + PBR^{-1}BPx = Qx + A^TPx.$$

Solution exists if we can find $P(t)$ satisfying

$$-\dot{P} = PA + A^TP - PBR^{-1}B^TP + Q \qquad P(T) = P_1$$

- This equation is called the *Riccati ODE*; matrix differential equation
- Can solve for $P(t)$ backwards in time and then apply $u(t) = -R^{-1}\, B\, P(t)\, x$
- Solving $x(t)$ forward in time gives optimal state (and input): $x^*(t)$, $u^*(t)$
- Note that $P(t)$ can be computed once (ahead of time) $\Rightarrow$ allows us to find the optimal trajectory from different points just by re-integrating state equation with optimal input

Finite Time LQR Summary



Problem: find trajectory that minimizes

$$\dot{x} = Ax + Bu \quad x = \mathbb{R}^n$$
$$x(0) \text{ given} \qquad u \in \mathbb{R}^p$$

$$\tilde{J} = \frac{1}{2}\int_0^T \left(x^T Q x + u^T R u\right)\, dt + \frac{1}{2} x^T(T) P_1 x(T)$$

Solution: time-varying linear feedback

$$u(t) = -R^{-1} B P(t) x.$$

$$-\dot{P} = PA + A^T P - PBR^{-1}B^T P + Q \qquad P(T) = P_1$$

- **Note: this is in feedback form $\Rightarrow$ can actually eliminate the controller (!)**

Infinite Time LQR

Extend horizon to $T = \infty$ and eliminate terminal constraint:

$$\dot{x} = Ax + Bu \quad x = \mathbb{R}^n \qquad x(0) \text{ given} \quad u \in \mathbb{R}^p$$

$$J = \int_0^\infty (x^T Q x + u^T R u)\, dt$$

Solution: same form, but can show P is *constant*

$$u = Kx \qquad K = -R^{-1} B^T P \quad \longleftarrow \text{State feedback (constant gain)}$$

$$0 = PA + A^T P - PBR^{-1}B^T P + Q \quad \longleftarrow \textit{Algebraic} \text{ Riccati equation}$$

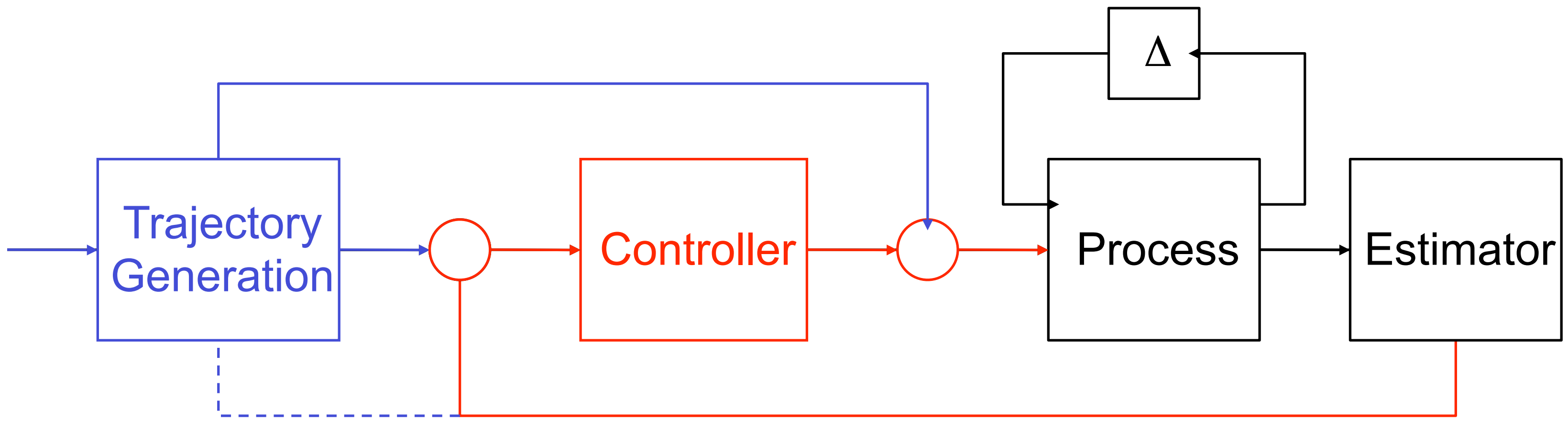
Remarks

- In MATLAB, `K = lqr(A, B, Q, R)`
- Require $R > 0$ but $Q \geq 0$ + must satisfy "observability" condition
- Alternative form: minimize "output" $y = H x$

$$L = \int_0^\infty x^T H^T H x + u^T R u\, dt = \int_0^\infty \|Hx\|^2 + u^T R u\, dt$$

- Require that (A, H) is observable. Intuition: if not, dynamics may not affect cost $\Rightarrow$ ill-posed. We will study this in more detail when we cover observers

Applying LQR Control



Application #1: trajectory generation

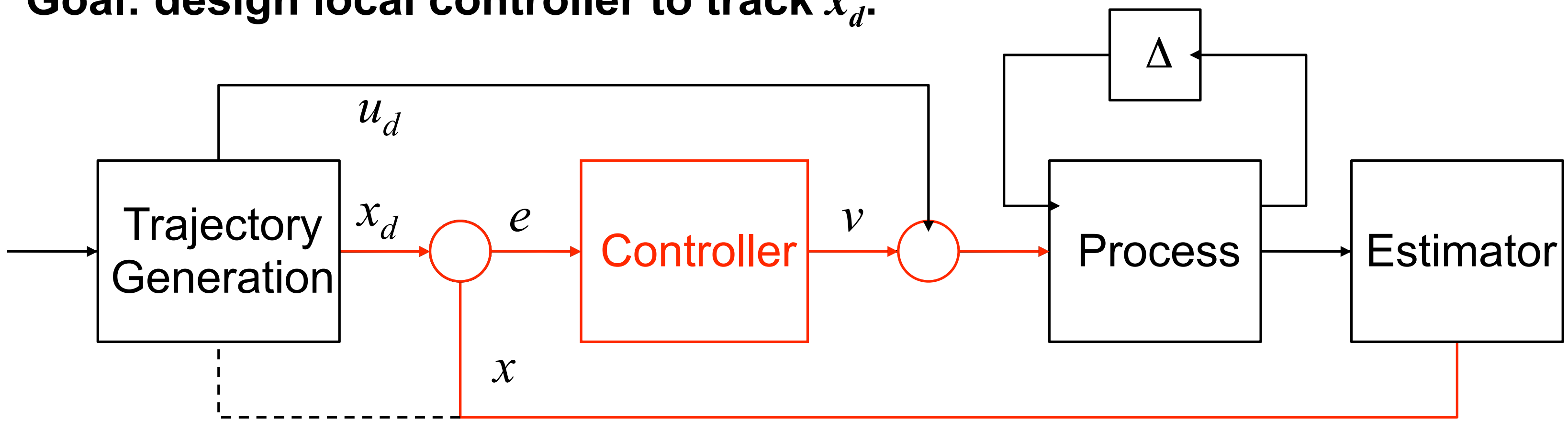
- Solve for (x_d, y_d) that minimize quadratic cost over finite horizon (requires linear process)
- Use local controller to regulate to desired trajectory

Application #2: trajectory tracking

- Solve LQR problem to stabilize the system to the origin $\Rightarrow$ *feedback* $u = K x$
- Can use this for local stabilization of *any* desired trajectory
- Missing: so far, have assumed we want to keep x small (versus $x \to x_d$)

LQR for trajectory tracking

Goal: design local controller to track x_d:



Approach: regulate the *error dynamics*

- Let $e = x - x_d$, $v = u - u_d$ and assume $f(x, u) = f(x) + g(x)\,u$ (simplifies notation)

$$
\begin{aligned}
\dot{e} = \dot{x} - \dot{x}_d &= f(x) + g(x)u - f(x_d) + g(x_d)u_d \\
&= f(e + x_d) - f(x_d) + g(e + x_d)(v + u_d) - g(x_d)u_d \\
&= F(e, v, x_d(t), u_d(t))
\end{aligned}
$$

- Now linearize the dynamics around $e = 0$ and design controller $v = K\,e$
- Final control law will be $u = K\,(x - x_d) + u_d$
- Note: in general, linearization will depend on $x_d \Rightarrow u = K(x_d)x$ $\leftarrow$ "gain scheduling"

Choosing LQR weights

Most common case: diagonal weights

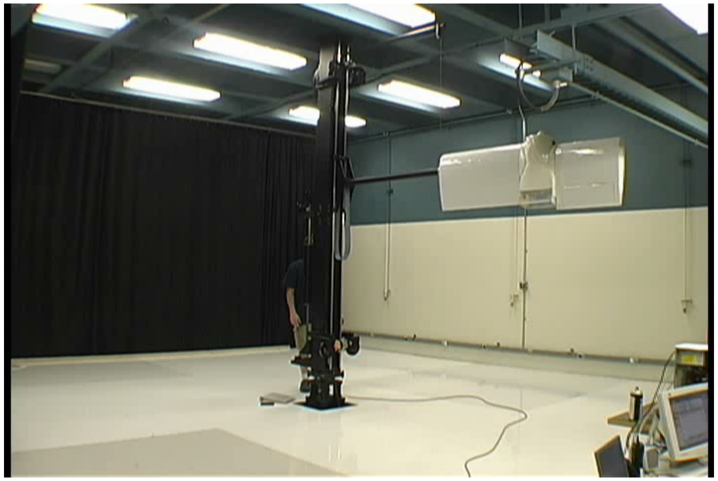
$$Q = \begin{bmatrix} q_1 & & \\ & \ddots & \\ & & q_n \end{bmatrix} \qquad R = \rho \begin{bmatrix} r_1 & & \\ & \ddots & \\ & & r_n \end{bmatrix}$$

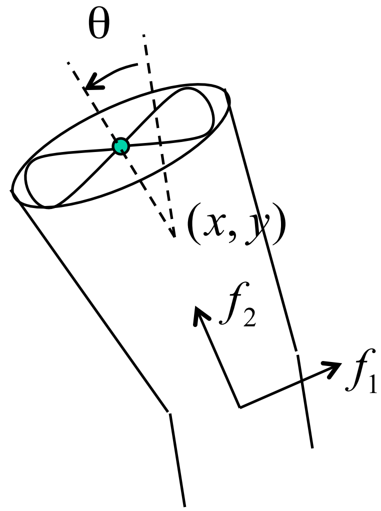
- Weight each state/input according to how much it contributes to cost
- Eg: if error in x_1 is 10x as bad as error in x_2, then $q_1 = 10\, q_2$
- OK to set some state weights to zero, but *all* input weights must be > 0
- Remember to take units into account: eg for ducted fan if position error is in meters and pitch error is in radians, weights will have different "units"

Remarks

- LQR will *always* give a stabilizing controller, but no gauranteed margins
- LQR shifts design problem from loop shaping to weight choices
- Most practical design uses LQR as a first cut, and then tune based on system performance

Example: Ducted Fan





Stabilization:
- Given an equilibrium position (x_d, y_d) and equilibrium thrust f_{2d}, maintain stable hover
- Full state available for feedback

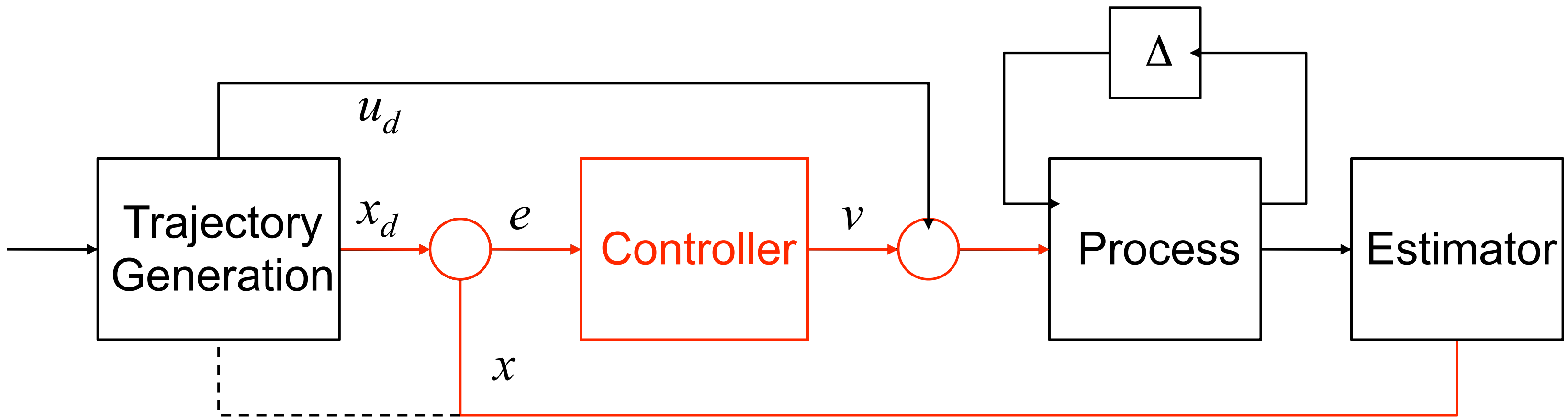
Tracking:
- Given a reference trajectory $(x_r(t), y_r(t))$, find a feasible trajectory $\vec{x}_d, u_d$ and a controller $u = \alpha(x, x_d, u_d)$ such that $x \to x_d$

Equations of motion

$$
\begin{aligned}
m\ddot{x} &= f_1 \cos\theta - f_2 \sin\theta - c_{d,x}(\theta, \dot{x}) \\
m\ddot{y} &= f_1 \sin\theta + f_2 \cos\theta - mg - c_{d,y}(\theta, \dot{y}) \\
J\ddot{\theta} &= r f_1 - mgl \sin\theta - c_{d,\theta}(\theta, \dot{\theta})
\end{aligned}
$$

LQR design: see lqr_dfan.m (available on course web page)

Variation: Integral Action



Limitation in LQR control: perfect tracking requires perfect model

- Control law is $u = K(x - x_d) + u_d$. $\Rightarrow u_d$ must be perfect to hold $e = 0$
- Alternative: use integral feedback to give zero steady state error

$$\frac{d}{dt}\begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} Ax + Bu \\ y - r \end{bmatrix} = \begin{bmatrix} Ax + Bu \\ Cx - r \end{bmatrix}$$

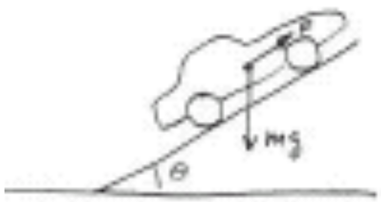
integral of (output) error

- Now design LQR controller for extended system (including integrator weight)

$$u = K(x - x_d) - K_i z + u_d$$

equilibrium value $\Rightarrow y = r \Rightarrow$ 0 steady state error

Example: Cruise Control



$$m\dot{v} = F - F_d$$
$$F = \alpha_n u T(\alpha_n v)$$
$$F_d = mgC_r + \frac{1}{2}\rho C_v A v^2 + mg\theta,$$

Linearized around v_0:

$$\dot{\tilde{v}} = a\tilde{v} - mg\theta + mb\tilde{u}$$
$$y = v = \tilde{v} + v_0$$

Step 1: augment linearized (error) dynamics with integrator

$$\frac{d}{dt}\begin{bmatrix}\tilde{v} \\ z\end{bmatrix} = \begin{bmatrix}\frac{a}{m} & 0 \\ -1 & 0\end{bmatrix}\begin{bmatrix}\tilde{v} \\ z\end{bmatrix} + \begin{bmatrix}b \\ 0\end{bmatrix} u + \begin{bmatrix}-g \\ 0\end{bmatrix}\theta + \begin{bmatrix}0 \\ r - v_0\end{bmatrix}$$

Step 2: choose LQR weights and compute LQR gains

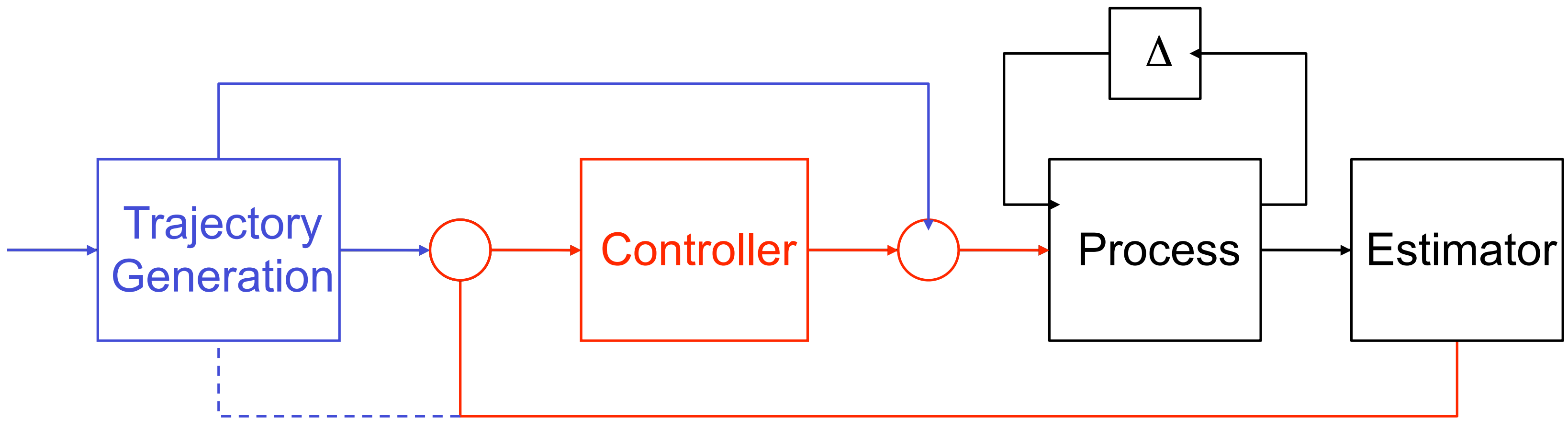
$$Q = \begin{bmatrix}q_1 & 0 \\ 0 & q_2\end{bmatrix} \qquad R = \rho \qquad \rightarrow \qquad K = \begin{bmatrix}k_1 & k_2\end{bmatrix}$$

Note: linearized about v_0 but try to maintain speed r (near v_0)

Step 3: implement controller

$$\dot{z} = v - r$$
$$u = u_0 + k_1(v - r) + k_2 z \quad \longleftarrow \quad \text{PI controller}$$

Summary: LQR Control



Application #1: trajectory generation

- Solve for (x_d, y_d) that minimize quadratic cost over finite horizon
- Use local controller to track trajectory

$$J = \frac{1}{2}\int_0^T \left(x^T Q x + u^T R u\right)\, dt + \frac{1}{2} x^T(T) P_1 x(T)$$

Application #2: trajectory tracking

- Solve LQR problem to stabilize the system
- Solve algebraic Riccati equation to get state gain
- Can augment to track trajectory; integral action

$$J = \int_0^\infty (x^T Q x + u^T R u)\, dt$$