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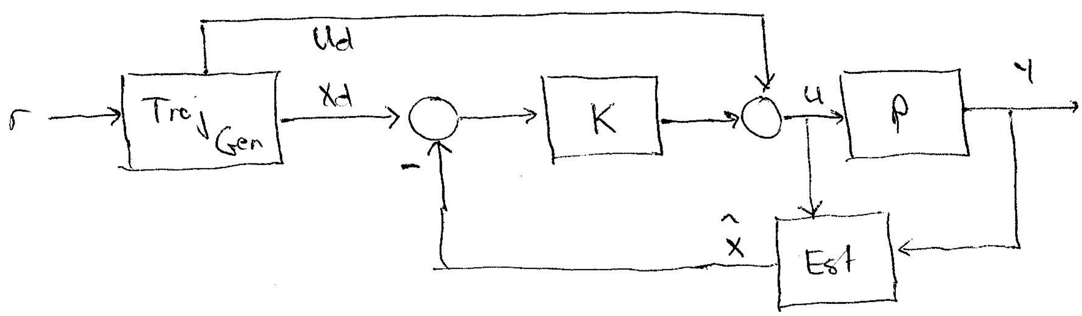
Trajectory tracking problem: Given a nonlinear control system

$$\dot{x} = f(x,u) \qquad x \in \mathbb{R}^n,\ u \in \mathbb{R}^p$$
$$y = h(x) \qquad y \in \mathbb{R}^q$$

and a reference trajectory $r(t) \in \mathbb{R}^q$, find a control law $u = \alpha(x, r)$ such that $\lim_{t\to\infty} (y(t) - r(t)) = 0$.

Approach: two degree of freedom design



Trajectory generation: find feasible trajectory (satisfies dynamics)

$$\left.\begin{aligned} \dot{x}_d &= f(x_d, u_d) \\ r &= h(x_d) \end{aligned} \quad \longrightarrow \quad x_d(r),\ u_d(r) \right\}$$

Will discuss this problem on Wed

Tracking: choose $u = \alpha(\hat{x}, x_d, u_d)$ such that closed loop system is stable, with desired performance

Estimation: determine $\hat{x}$ from y, u

Fall 2008: skip this page; will be covered in homework set #1. See AM08 for derivation of k_r.

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Review: reference tracking for linear systems

$$\dot{x} = Ax + Bu \qquad u = -Kx + k_r r$$
$$y = Cx \qquad r = \underline{\text{constant}}$$

1. Choose K to give desired closed loop dynamics (eigenvalue placement, LQR, etc)

2. Choose k_r to give desired output value at equilibrium

$$\dot{x} = Ax + Bu = (A - BK)x + Bk_r r$$

Eq pt: $x_e = -(A-BK)^{-1} Bk_r r$

$$y_e = -C(A-BK)^{-1} B k_r r \overset{\text{desired}}{=} r$$

$$\Rightarrow \quad k_r = -\frac{1}{C(A-BK)^{-1}B}$$

Remarks

1. This approach works by stabilizing the origin and then using $k_r r$ to "push" the system to x_e. Works because system is linear

2. Alternative formulation: solve directly for eq pt

$$\begin{bmatrix} 0 \\ r \end{bmatrix} = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \begin{bmatrix} x_d \\ u_d \end{bmatrix} \longrightarrow u = \underbrace{-K(x - x_d)}_{\text{stabilize eq pt}} + \underbrace{u_d}_{\text{nominal input}}$$

HW: show this is equiv to $k_r r$

3. Fundamental assumption: (x_d, u_d) is $\underline{\text{equil pt}}$ $\Rightarrow$ r is constant.

Gain scheduling

Nonlinear system with feasible trajectory:

$$\dot{x} = f(x, u) \qquad \dot{x}_d = f(x_d, u_d)$$
$$y = h(x) \qquad r(t) = h(x_d)$$

} Solve via optimal control or other means (more on Wed)

To stabilize the reference trajectory, look at error $e = x - x_d$

$$\dot{x} = f(x, u) \qquad e = x - x_d$$
$$\dot{x}_d = f(x_d, u_d) \qquad v = u - u_d$$

$$\dot{e} = f(x, u) - f(x_d, u_d) =: F(e, v, x_d, u_d)$$

$$= f(e + x_d, v + u_d) - f(x_d, u_d)$$

Nonlinear, time-varying system

We can now treat (x_d, u_d) as parameters in the controller and linearize around $(e, v) = (0, 0)$

$$\dot{e} \approx A_d e + B_d v \qquad A_d = \left.\frac{\partial F}{\partial e}\right|_{(0,0)} = \left.\frac{\partial f}{\partial x}\right|_{(x_d, u_d)}$$

$$B_d = \left.\frac{\partial F}{\partial v}\right|_{(0,0)} = \left.\frac{\partial f}{\partial u}\right|_{(x_d, u_d)}$$

Now stabilize $e = 0$ by choosing K_d such that $(A_d - B_d K_d)$ stable.

$$u = -K_d e + u_d = -K_d (x - x_d) + u_d$$

Note that K_d depends on (x_d, u_d). Eq

$$u = -K_d(x_d, u_d) \cdot (x - x_d) + u_d$$

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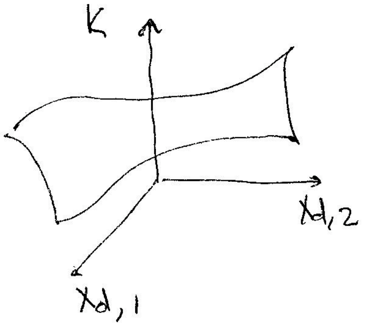
Remarks

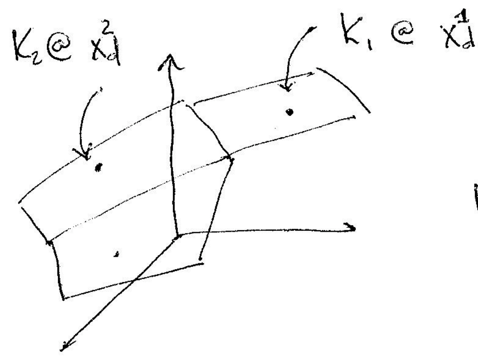
1. Does not assume (x_d, u_d) are equilibrium values, just that they satisfy dynamics. Eq pt is a special case (r = constant)

2. More generally, can schedule gains on any parameter, or even the state: $u = -K(x,\mu)\cdot(x - x_d) + u_d$. Use with care (NL)

3. Problem: linearizing about desired eq pt $\Rightarrow$ linearization may not work well if you are far from this point.

4. In practice, implement gain scheduling through interpolation





$$K_d = \sum \alpha_i(x) K_i$$

interpolation fcns

Sample implementation: "falcon" library (RMM web page)

5. Theory: not much can be said about when this works
   - OK if (x_d, u_d) are varying sufficiently slowly
   - Works well in practice, even if they aren't

Fall 2008: cover this example if there is time, otherwise can discuss on Friday

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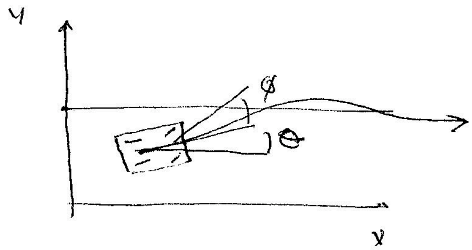
Example: steering control with velocity scheduling



$$\dot{x} = \cos\theta\, v \qquad \dot{y} = \sin\theta\, v \qquad \dot{\theta} = \frac{v}{l}\tan\phi$$

$$\left.\begin{aligned} x_d &= v_r t \\ y_d &= y_{ref} \\ \theta_d &= 0 \end{aligned}\right] x_d \qquad \left.\begin{aligned} v_d &= v_r \\ \phi_d &= 0 \end{aligned}\right] u_d$$

Note that $(x_d, y_d, \theta_d; v_d, \phi_d)$ are not at equilibrium, but they do satisfy the equations of motion.

$$A_d = \left.\frac{\partial f}{\partial x}\right|_{(x_d,u_d)} = \left.\begin{bmatrix} 0 & 0 & -\sin\theta \\ 0 & 0 & \cos\theta \\ 0 & 0 & 0 \end{bmatrix}\right|_{(x_d,u_d)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$B_d = \left.\frac{\partial f}{\partial u}\right|_{(x_d,u_d)} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & v_r/l \end{bmatrix} \longleftarrow \text{depends on } u_d \text{ through } v_r$$

Error dynamics: $e = x - x_d \qquad w = u - u_d$

$$\begin{aligned} \dot{e}_x &= w_1 \\ \dot{e}_y &= e_\theta \\ \dot{e}_\theta &= \frac{v_r}{l} w_2 \end{aligned} \qquad \left.\begin{aligned} w_1 &= -\lambda_1 e_x \\ w_2 &= -\frac{l}{v_r}(a_1 e_y + a_2 e_\theta) \end{aligned}\right\} \begin{aligned} &\text{pole at } \lambda_1 \text{ and} \\ &s^2 + a_1 s + a_2 = 0 \end{aligned}$$

Controller in original coordinates

$$\begin{bmatrix} v \\ \phi \end{bmatrix} = -\underbrace{\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \frac{a_1 l}{v_r} & \frac{a_2 l}{v_r} \end{bmatrix}}_{K_d} \underbrace{\begin{bmatrix} x - v_r t \\ y - y_r \\ \theta \end{bmatrix}}_{e} + \underbrace{\begin{bmatrix} v_r \\ 0 \end{bmatrix}}_{u_d}$$

Intuition: at slower speeds, turn wheel harder to get same time response. Note that gains blow up if $v_r = 0$ (not ...)