CALIFORNIA INSTITUTE OF TECHNOLOGY
Control and Dynamical Systems

CDS 110b

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Optimal Control

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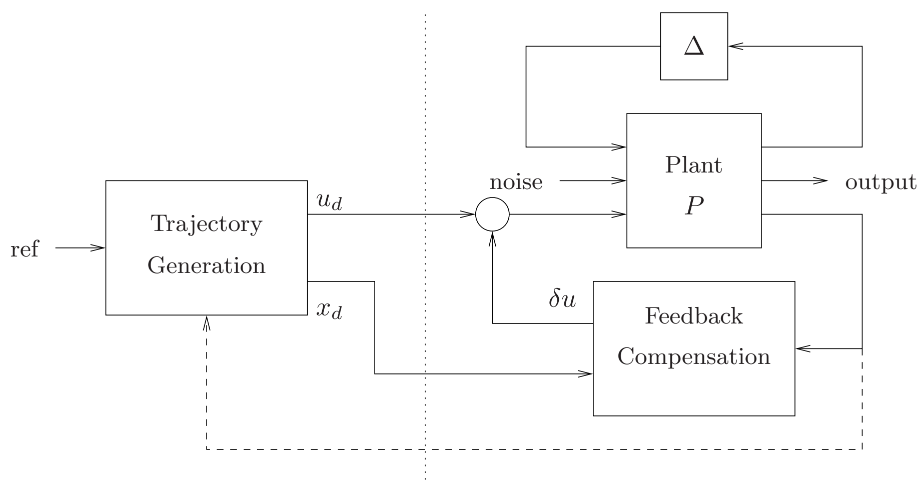
This lecture provides an overview of optimal control theory. Beginning with a review of optimization, we introduce the notion of Lagrange multipliers and provide a summary of the Pontryagin's maximum principle.

1 Introduction: Optimization-Based Control

Basic idea: two degree of freedom control design:

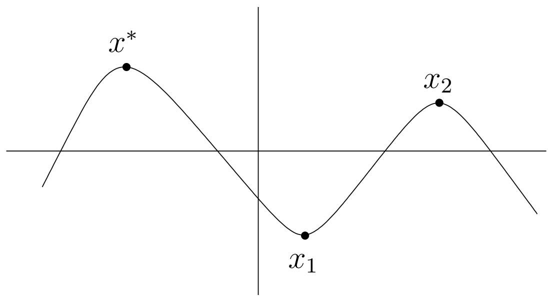


- Compute feasiable trajectory (x_d, u_d) for the system
- Use local controller to track feasible trajectory

Variation (later): update the trajectory based on the current state of the system $\implies$ "receding horizon control".

2 Review: Optimization

Optimization of *functions*: given $F : \mathbb{R}^n \to \mathbb{R}$, find $x^* \in \mathbb{R}^n$ such that $F(x^*) \geq F(x)$ for all $x \in \mathbb{R}^n$.



Necessary conditions:

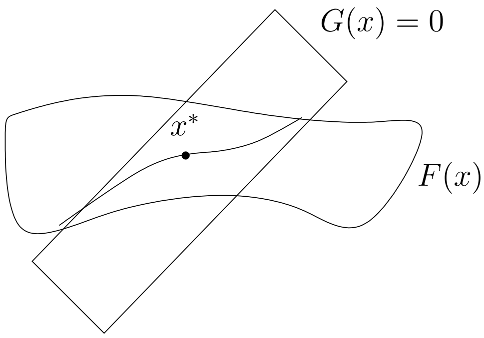
$$\frac{\partial F}{\partial x}(x^*) = 0$$

These are *not* sufficient conditions; the points x_1 and x_2 and x^* all satisfy the necessary condition but only one is the (global) maximum.

Optimization with *constraints*: given $F : \mathbb{R}^n \to \mathbb{R}$ and $G_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, k$, find $x^* \in \mathbb{R}^n$ such that $G_i(x^*) = 0$ (satisfies constraints) and

$$F(x^*) \geq F(x)$$

for all x such that $G_i(x) = 0$.



Necessary conditions: $\exists\ \lambda_i$, $i = 1, \ldots, k$ such that

$$\frac{\partial F}{\partial x}(x^*) + \sum_{i=1}^{k} \lambda_i \frac{\partial G_i}{\partial x}(x^*) = 0$$

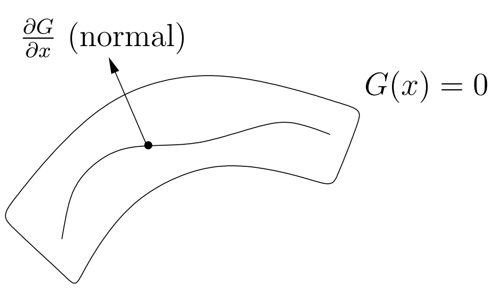
or define $\widetilde{F} = F + \sum \lambda_i G_i \implies$

$$\frac{\partial \widetilde{F}}{\partial x}(x^*) = 0$$

The scalars λ_i are called *Lagrange multipliers*.

Intuition on Lagrange multiplers

View #1 (geometric):



In the expression

$$\frac{\partial F}{\partial x} + \lambda^T \frac{\partial G}{\partial x} = 0$$

the term $\frac{\partial F}{\partial x}$ is the usual (gradient) optimality condition while the term $\frac{\partial G}{\partial x}$ is used to "cancel" the gradient in the directions normal to the constraint.

View #2 (algebraic): In the expression

$$\min_x F(x) + \lambda^T G(x)$$

the variables λ can be regarded as free variables $\implies$ need to choose x such that $G(x) = 0$. Otherwise, we can choose λ to generate a large cost.

Remarks:

1. We will often switch back and forth betweeen max and min:

$$\max_x F(x) = \min_x \bigl(-F(x)\bigr)$$

2. Very good software is available for solving optimization problems numerically: NPSOL, SNOPT, fmin (in MATLAB).

3 Optimal Control of Systems

Consider now the *optimal control problem*:

$$\min_u \underbrace{\int_0^T L(x,u)\,dt}_{\text{integrated cost}} + \underbrace{V\bigl(x(T),u(T)\bigr)}_{\text{final cost}}$$

subject to the constraint

$$\dot{x} = f(x,u) \qquad x \in \mathbb{R}^n,\ u \in \mathbb{R}^m$$

This problem is equivalent to the "standard" problem of minimizing a cost function $J(z)$ where $z \in L_2[0,T]$ and $h(z) = 0$ models the dynamics.

Basic idea: optimize over *curves* $x(t)$, $u(t)$ that satisfy the *constraint* $\dot{x}(t) - f(x(t),u(t)) = 0$.

Remarks

1. A special case is *linear quadratic* optimizal control:

$$\dot{x} = Ax + Bu \qquad J = \tfrac{1}{2}\int_0^T \left(x^T Q x + u^T R u\right)\,dt + x^T(T) P_1 x(T)$$

$$\begin{aligned} Q \geq 0 &\quad \text{penalizes state error (assumes } x_d = 0) \\ R > 0 &\quad \text{penalizes input (\textit{must} be positive definite)} \\ P_1 > 0 &\quad \text{penalizes terminal state} \end{aligned}$$

2. Variation: let $T = \infty$, $V = 0$ $\implies$ "infinite horizon" problem

3. Use of optimal control does not require linear systems or quadratic cost. *Very* general results are available.

Maximum Principle (Pontryagin, 1960s)

System: nonlinear ODE

$$\dot{x} = f(x,u) \quad x = \mathbb{R}^n$$
$$x(0) \text{ given} \quad u \in \Omega \subset \mathbb{R}^p$$

where $f(x,u) = (f_1(x,u), \ldots f_n(x,u)) : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$.

Cost: integral + terminal cost and constraint

$$J = \int_0^T L(x,u)\,dt + V(x(T))$$
$$\psi(x(T)) = 0 \quad \leftarrow \text{terminal constraint}$$

Hamiltonian:

$$H = L + \lambda^T f = L + \sum \lambda_i f_i$$

Theorem 1. *If (x^*, u^*) is optimal, then $\exists\ \lambda^*(t)$ and ν^* such that*

$$\dot{x}_i = \frac{\partial H}{\partial \lambda_i} \qquad -\dot{\lambda}_i = \frac{\partial H}{\partial x_i} \qquad \underbrace{\begin{array}{c} x(0) \textit{ given}, \ \psi(x(T)) = 0 \\ \lambda(T) = \frac{\partial V}{\partial x}(x(T)) + \frac{\partial \psi^T}{\partial x}\nu \end{array}}_{\text{Boundary conditions } (2n \text{ total})}$$

and

$$H(x^*(t), u^*(t), \lambda^*(t)) \leq H(x^*(t), u, \lambda^*(t)) \quad \forall \quad u \in \Omega$$

Remarks

1. Note that even though we are *minimizing* the cost, this is still usually called the maximum principle (artifact of history).
2. This is a *very* general (and elegant) theorem. Generalizes optimization to optimal control.
3. If $u = \text{argmin} H(x,u,\lambda)$ exists, we can use this to *choose* control law u
4. If $\Omega = \mathbb{R}^m$ (unconstrained input) and H is differentiable, then a necessary condition is
$$\frac{\partial H}{\partial u} = 0$$
5. Proof: Lewis and Syrmos [1]

4 Examples

Scalar linear system Consider the optimal control problem for the system

$$\dot{x} = ax + bu, \tag{1}$$

where $x = \mathbb{R}$ is a scalar state, $u \in \mathbb{R}$ is the input, the initial state $x(t_0)$ is given, and $a, b \in \mathbb{R}$ are positive constants. We wish to find a trajectory $(x(t), u(t))$ that minimizes the cost function

$$J = \tfrac{1}{2} \int_{t_0}^{t_f} u^2(t)\, dt + \tfrac{1}{2} c x^2(t_f),$$

where the terminal time t_f is given and $c > 0$ is a constant. This cost function balances the final value of the state with the input required to get to that position.

To solve the problem, we define the various elements used in the maximum principle. Our integrated and terminal costs are given by

$$L = \tfrac{1}{2} u^2(t) \qquad V = \tfrac{1}{2} c x^2(t_f).$$

We write the Hamiltonian of this system and derive the following expressions:

$$\begin{aligned} H &= L + \lambda f = \tfrac{1}{2} u^2 + \lambda(ax + bu) \\ \dot{\lambda} &= -\frac{\partial H}{\partial x} = -a\lambda \\ \lambda(t_f) &= \frac{\partial V}{\partial x} = c x(t_f) \\ \Rightarrow \lambda(t) &= c x(t_f) e^{a(t_f - t)} \end{aligned}$$

The optimal control is given by

$$\frac{\partial H}{\partial u} = u + b\lambda = 0 \Rightarrow u^*(t) = -b\lambda(t) = -bcx(t_f)e^{a(t_f - t)}.$$

Substituting this control into the dynamics given by equation (1) yields a first-order ODE in x

$$\dot{x} = ax - b^2 c x(t_f) e^{a(t_f - t)}$$

that can be solved explicitly as

$$x^*(t) = x(t_o) e^{a(t - t_o)} + \frac{b^2 c}{2a} x^*(t_f) \left[e^{a(t_f - t)} - e^{a(t + t_f - 2t_o)} \right].$$

Setting $t = t_f$ and solving for $x(t_f)$ gives

$$x^*(t_f) = \frac{2a\, e^{a(t_f - t_o)} x(t_o)}{2a - b^2 c \left(1 - e^{2a(t_f - t_o)}\right)}$$

and finally we can write

$$u^*(t) = -\frac{2abc\, e^{a(2t_f - t_o - t)} x(t_o)}{2a - b^2 c \left(1 - e^{2a(t_f - t_o)}\right)}$$

$$x^*(t) = x(t_o) e^{a(t - t_o)} + \frac{b^2 c\, e^{a(t_f - t_o)} x(t_o)}{2a - b^2 c \left(1 - e^{2a(t_f - t_o)}\right)} \left[e^{a(t_f - t)} - e^{a(t + t_f - 2t_o)}\right].$$

We can use the form of this expression to explore how our cost function affects the optimal trajectory. For example, we can ask what happens to the terminal state $x^*(t_f$ and $c \to \infty$. Setting $t = t_f$ in equation (4) and taking the limit we find that

$$\lim_{c \to \infty} x^*(t_f) = 0.$$

Bang-bang control

Problem specification:

$$\begin{aligned} &\dot{x} = Ax + Bu \qquad |u| \leq 1 \\ &J = \int_0^T 1\, dt \qquad\qquad \longleftarrow \text{minimum } \textit{time} \\ &\psi(x(T)) = x(T) \qquad \longleftarrow \text{move to origin} \end{aligned}$$

Form the Hamiltonian:

$$\begin{aligned} H &= 1 + \lambda^T (Ax + Bu) \\ &= 1 + (\lambda^T A)x + (\lambda^T B)u \end{aligned}$$

Now apply the conditions in the maximum principle:

$$\begin{aligned} \dot{x} &= \frac{\partial H}{\partial \lambda} = Ax + Bu \\ -\dot{\lambda} &= \frac{\partial H}{\partial x} = A^T \lambda \\ u &= \text{arg min } H = -\text{sgn}(\lambda^T B) \end{aligned}$$

The optimal solution always satisfies this equation (necessary condition) with $x(0) = x_0$ and $x_(T) = 0$. It follows that the input is always $u = \pm 1 \implies$ “bang-bang”.

References

[1] F. L. Lewis and V. L. Syrmos. *Optimal Control*. Wiley, second edition, 1995.