CALIFORNIA INSTITUTE OF TECHNOLOGY
Control and Dynamical Systems

CDS 110

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Winter 2007 | | Due: 9 Mar 07 (Fri)

1. Consider the following linear system

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 1 \\ 1 \end{bmatrix} v$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + w$$

where v and w are Gaussian white noise with covariances $r > 0$ and 1, respectively. Suppose we wish to design a controller that minimizes the cost function

$$J = \int_0^\infty q x^T \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} x + u^2 dt$$

where $q > 0$.

(a) Design a controller for the system using a Kalman filter and optimal linear quadratic regulator. Give the transfer function for the resulting compensator.

(b) Show that the resulting closed loop system has vanishingly small gain margin for r and q chosen sufficiently large. (Hint: you should spend about 30 minutes trying this problem and then go and read the 1976 paper by Doyle if you can't get it. Be careful about differences in notation between the paper and the problem statement.)

2. Consider the class of perturbed plants of the form

$$\frac{P}{1 + \Delta W_2 P},$$

where W_2 is a fixed stable weighting function with W_2 strictly proper and Δ is a variable stable transfer function with $\|\Delta\|_\infty \leq 1$. Assume that C is a controller achieving stability for P. Prove that C provides internal stability for the perturbed plant if $\|W_2 P S\|_\infty < 1$.

Students who are not doing the course project should complete the following additional problems:

3. This problem shows that the stability margin is critically dependent on the type of perturbation. The setup is a unity-feedback loop with controller $K(s) = 1$ and plant $P_{nom}(s) + \Delta(s)$, where

$$P_{nom}(s) = \frac{10}{s^2 + 0.2s + 1}$$

(a) Assume $\Delta(s)$ is a stable transfer function. Compute the largest β such that the feedback system is internally stable for all $\|\Delta\|_\infty < \beta$.

(b) Repeat but with $\Delta \in \mathbb{R}$.

4. Consider the following model for the pitch dynamics of the Caltech ducted fan:

$$P(s) = \frac{r}{Js^2 + bs + mgl} \qquad \begin{array}{lll} g = 9.8 \text{ m/sec}^2 & m = 1.5 \text{ kg} & b = 0.05 \text{ kg/sec} \\ l = 0.05 \text{ m} & J = 0.0475 \text{ kg m}^2 & r = 0.25 \text{ m} \end{array}$$

We wish to design a robust controller that satisfies the following: performance specification:

- Steady state error of less than 1%
- Tracking error of less than 5% from 0 to 1 Hz (remember to convert this to rad/sec).

(a) Write the above specification as a weighted sensitivity specification. Give explicit formulas for the frequency weight(s).

(b) Consider a plant perturbation of 20% variation in the value of r around the nominal value. Design a controller that provides robust stabilty with respect to this perturbation. (You can use any technique to design this controller, but you might try designing an estimator + state feedback controller as a first cut.)