CALIFORNIA INSTITUTE OF TECHNOLOGY
Control and Dynamical Systems

CDS 110b

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1. Consider the optimal control problem for the system

$$\dot{x} = ax + bu \qquad J = \tfrac{1}{2}\int_{t_0}^{t_f} u^2(t)\,dt + \tfrac{1}{2}cx^2(t_f),$$

where $x \in \mathbb{R}$ is a scalar state, $u \in \mathbb{R}$ is the input, the initial state $x(t_0)$ is given, and $a, b \in \mathbb{R}$ are positive constants. We take the terminal time t_f as given and let $c > 0$ be a constant that balances the final value of the state with the input required to get to that position. The optimal is derived in the lecture notes for week 6 and is shown to be

$$\begin{aligned} u^*(t) &= -\frac{2abc\, e^{a(2t_f - t_o - t)} x(t_o)}{2a - b^2 c\left(1 - e^{2a(t_f - t_o)}\right)} \\ x^*(t) &= x(t_o) e^{a(t-t_o)} + \frac{b^2 c\, e^{a(t_f - t_o)} x(t_o)}{2a - b^2 c\left(1 - e^{2a(t_f - t_o)}\right)} \left[e^{a(t_f - t)} - e^{a(t + t_f - 2t_o)} \right]. \end{aligned} \tag{1}$$

Now consider the infinite horizon cost

$$J = \tfrac{1}{2}\int_{t_0}^{\infty} u^2(t)\,dt$$

with $x(t)$ at $t = \infty$ constrained to be zero.

(a) Solve for $u^*(t) = -bPx^*(t)$ where P is the positive solution corresponding to the algebraic Riccati equation. Note that this gives an explicit feedback law ($u = -bPx$).

(b) Plot the state solution of the finite time optimal controller for the following parameter values

$$\begin{gathered} a = 2 \qquad b = 0.5 \qquad x(t_0) = 4 \\ c = 0.1,\ 10 \qquad t_f = 0.5,\ 1,\ 10 \end{gathered}$$

(This should give you a total of 6 curves.) Compare these to the infinite time optimal control solution. Which finite time solution is closest to the infinite time solution? Why?

2. Consider a nonlinear control system

$$\dot{x} = f(x, u)$$

with linearization

$$\dot{x} = Ax + Bu.$$

Show that if the linearized system is reachable, then there exists a (local) control Lyapunov function for the nonlinear system. (Hint: use the solution to the LQR optimal control problem for the linearized system.)

Students who are not doing the course project should complete the following additional problems:

3. Using the solution given in equation (1), implement the finite-time optimal controller in a receding horizon fashion with an update time of $\delta = 0.5$. Using the parameter values in problem 1(b), Compare the responses of the receding horizon controllers to the LQR controller you designed for problem 1, from the same initial condition. What do you observe as c and t_f increase?

   (Hint: you can write a MATLAB script to do this by performing the following steps:

   (i) set $t_0 = 0$

   (ii) using the closed form solution for x^* from problem 1, plot $x(t)$, $t \in [t_0, t_f]$ and save $x_\delta = x(t_0 + \delta)$

   (iii) set $x(t_0) = x_\delta$ and repeat step (ii) until x is small.)

4. In this problem we will explore the effect of constraints on control of the linear unstable system given by

$$\dot{x}_1 = 0.8x_1 - 0.5x_2 + 0.5u$$
$$\dot{x}_2 = x_1 + 0.5u$$

   subject to the constraint that $|u| \leq a$ where a is a postive constant.

   (a) Ignore the constraint ($a = \infty$) and design an LQR controller to stabilize the system. Plot the response of the closed system from the initial condition given by $x = (1, 0)$.

   (b) Use SIMULINK or `ode45` to simulate the the system for some finite value of a with an initial condition $x(0) = (1, 0)$. Numerically (trial and error) determine the smallest value of a for which the system goes unstable.

   (c) Let $a_{\min}(\rho)$ be the smallest value of a for which the system is unstable from $x(0) = (\rho, 0)$. Plot $a_{\min}(\rho)$ for $\rho = 1, 4, 16, 64, 256$.

   (d) *Optional*: Given $a > 0$, design and implement a receding horizon control law for this system. Show that this controller has larger region of attraction than the controller designed in part (b). (Hint: solve the finite horizon LQ problem analytically, using the bang-bang example as a guide to handle the input constraint.)