CALIFORNIA INSTITUTE OF TECHNOLOGY
Control and Dynamical Systems

CDS 110b

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Problem Set #5 (rev 1)

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1. In this problem, you will use the maximum principle to show that the shortest path between two points is a straight line.

   We model the problem by constructing a control system

   $$\dot{x} = u$$

   where $x \in \mathbb{R}^2$ is the position in the plane and $u \in \mathbb{R}^2$ is the velocity vector along the curve. Suppose we wish to find a curve of minimal length connecting $x(0) = x_0$ and $x(1) = x_f$. To minimize the length, we minimize the integral of the velocity along the curve,

   $$J = \int_0^1 \sqrt{\|\dot{x}\|}\, dt,$$

   subject to to the initial and final state constraints. Use the maximum principle to show that the minimal length path is indeed a straight line at maximum velocity. (Hint: minimizing $\sqrt{\|\dot{x}\|}$ is the same as minimizing $\dot{x}^T \dot{x}$; this will simplify the algebra a bit.)

2. Consider a linear system with input u and output y and suppose we wish to minimize the quadratic cost function

   $$J = \int_0^\infty \left(y^T y + \rho u^T u\right)\, dt.$$

   Show that if the corresponding linear system is observable, then the closed loop system obtained by using the optimal feedback $u = -Kx$ is guaranteed to be stable.

Students who are not doing the course project should complete the following additional problems:

3. (Friedland 2.1, 3.6, 7.2, 9.10) Consider the motor-driven inverted pendulum on a cart, whose linearized dynamics are given by

   $$\ddot{x} + \frac{k^2}{Mr^2R}\dot{x} + \frac{mg}{M}\theta = \frac{k}{MRr}u$$
   $$\ddot{\theta} - \left(\frac{M+m}{Ml}\right) g\theta - \frac{k^2}{Mr^2Rl}\dot{x} = -\frac{k}{MRrl}u$$

   where k is the motor torque constant, R is the motor resistance, r is the ratio of the linear forces applied to the cart ($\tau = rf$), and u is the voltage applied to the motor. The following numerical data may be used:

   $$m = 0.1 \text{ kg} \qquad M = 1.0 \text{ kg} \qquad l = 1.0 \text{ m} \qquad g = 9.8 \text{ m/s}^2$$
   $$k = 1 \text{ V}\cdot\text{s} \qquad R = 100\ \Omega \qquad r = 0.02 \text{ m}$$

We wish to optimize the gains using a performance criterion of the form

$$V = \int_0^\infty \left(q_1^2 x^2 + q_3^2 \theta^2 + \rho^2 u^2 \right) \, dt$$

A pendulum angle much greater than 1 degree = 0.017 rad would be precarious. Thus a heavy weighting error on θ is indicated: $q_3^2 = 1/(0.017)^2 \approx 3000$. For the physical dimensions of the system, a position error of the order of 10 cm = 0.1 m is not unreasonable. Hence $q_1^2 = 1/(0.1)^2 = 100$.

(a) Using these values of q_1^2 and q_3^2, determine and plot the gain matrices and corresponding closed loop poles as a function of the control weighting parameter ρ^2 for $0.001 < \rho^2 < 50$.

(b) Repeat part (a) for a heavier weighting: $q_1^2 = 10^4$ on the cart displacement.

(c) Plot the step responses for the controllers you defined in parts (a) and (b) and explain their behavior in terms of the cost functions you used.

4. Consider the problem of moving a two-wheeled mobile robot (eg, a Segway) from one position and orientation to another. The dynamics for the system is given by the nonlinear differential equation

$$\dot{x} = \cos\theta \, v$$
$$\dot{y} = \sin\theta \, v$$
$$\dot{\theta} = \omega$$

where (x, y) is the position of the rear wheels, θ is the angle of the robot with respect to the x axis, v is the forward velocity of the robot and ω is spinning rate. We wish to choose an input (v, ω) that minimizes the time that it takes to move between two configurations (x_0, y_0, θ_0) and (x_f, y_f, θ_f), subject to input constraints $|v| \leq L$ and $|\omega| \leq M$.

Use the maximum principle to show that any optimal trajectory consists of segments in which the robot is traveling at maximum velocity in either the forward or reverse direction, and going either straight, hard left ($\omega = -M$) or hard right ($\omega = +M$).

Note: one of the cases is a bit tricky and can't be completely proven with the tools we have learned so far. However, you should be able to show the other cases and verify that the tricky case is possible.