CALIFORNIA INSTITUTE OF TECHNOLOGY
Control and Dynamical Systems

CDS 110b

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Unless otherwise specified, you may use MATLAB or Mathematica as long as you include a copy of the code used to generate your answer.

1. A random variable Y is the sum of two independent normally (Gaussian) distributed random variables having means m_1, m_2 and variances σ_1^2, σ_2^2 respectively. Show that the probability density function for Y is

$$p(y) = \frac{1}{2\pi\sigma_1\sigma_2}\int_{-\infty}^{\infty} \exp\left\{-\frac{(y-x-m_1)^2}{2\sigma_1^2} - \frac{(x-m_2)^2}{2\sigma_2^2}\right\} dx$$

and confirm that this is normal (Gaussian) with mean $m_1 + m_2$ and variance $\sigma_1^2 + \sigma_2^2$. (Hint: most of the definitions you need should be in the notes posted on the web.)

2. Consider a second order system with dynamics

$$\begin{bmatrix}\dot x_1 \\ \dot x_2\end{bmatrix} = \begin{bmatrix}-a & 0 \\ 0 & -b\end{bmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix} + \begin{bmatrix}1 \\ 1\end{bmatrix} v$$
$$y = \begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix}$$

that is forced by Gaussian white noise with zero mean and variance σ^2. Assume $a, b > 0$.

   (a) Compute the correlation function $\rho(\tau)$ for the output of the system. Your answer should be an explicit formula in terms of a, b and σ.

   (b) Assuming that the input transients have died out, compute the mean and variance of the output.

3. (Friedland 10.1) Consider a system with transfer function

$$H(s) = \frac{1}{(s+\alpha)(s+\beta)}.$$

Assume that the input to the system is white noise with a spectral density of unity.

   (a) Find the spectral density $S(\omega)$

   (b) Find the mean square of the output by computing $\rho(0)$ using the inverse Fourier transform of $S(\omega)$ evaluated at $\tau = 0$.

   (c) Write a state space realization for $H(s)$ and compute the mean square of the output Y solving the appropriate Lyapunov equation.

4. Find a constant matrix A and vectors F and C such that for

$$\dot{x} = Ax + Fw, \; y = Cx$$

the power spectrum of y is given by

$$S(\omega) = \frac{1+\omega^2}{(1-7\omega^2)^2+1}$$

Describe the sense in which your answer is unique.