1. A random variable $Y$ is the sum of two independent normally (Gaussian) distributed random variables having means $m_1, m_2$ and variances $\sigma_1^2, \sigma_2^2$ respectively. Show that the probability density function for $Y$ is

$$p(y) = \frac{1}{2\pi\sigma_1\sigma_2} \int_{-\infty}^{\infty} \exp \left\{ -\frac{(y-x-m_1)^2}{2\sigma_1^2} - \frac{(x-m_2)^2}{2\sigma_2^2} \right\} dx$$

and confirm that this is normal (Gaussian) with mean $m_1 + m_2$ and variance $\sigma_1^2 + \sigma_2^2$. (Hint: most of the definitions you need should be in the notes posted on the web.)

2. Consider a second order system with dynamics

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -a & 0 \\ 0 & -b \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} v$$

$$y = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

that is forced by Gaussian white noise with zero mean and variance $\sigma^2$. Assume $a, b > 0$.

(a) Compute the correlation function $\rho(\tau)$ for the output of the system. Your answer should be an explicit formula in terms of $a, b$ and $\sigma$.

(b) Assuming that the input transients have died out, compute the mean and variance of the output.

3. (Friedland 10.1) Consider a system with transfer function

$$H(s) = \frac{1}{(s + \alpha)(s + \beta)}.$$ 

Assume that the input to the system is white noise with a spectral density of unity.

(a) Find the spectral density $S(\omega)$

(b) Find the mean square of the output by computing $\rho(0)$ using the inverse Fourier transform of $S(\omega)$ evaluated at $\tau = 0$.

(c) Write a state space realization for $H(s)$ and compute the mean square of the output $Y$ solving the appropriate Lyapunov equation.
4. Find a constant matrix $A$ and vectors $F$ and $C$ such that for

$$\dot{x} = Ax + Fw, \ y = Cx$$

the power spectrum of $y$ is given by

$$S(\omega) = \frac{1 + \omega^2}{(1 - 7\omega^2)^2 + 1}$$

Describe the sense in which your answer is unique.