CALIFORNIA INSTITUTE OF TECHNOLOGY
Control and Dynamical Systems

CDS 110b

Problem Set #1

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Unless otherwise specified, you may use MATLAB or Mathematica as long as you include a copy of the code used to generate your answer.

1. The linearized equations of motion of a simple pendulum are given by

$$\ddot{\theta} + \omega_0^2 \theta = u$$

   (a) Design a state feedback controller so that the roots of the closed-loop characteristic equation are at $s = -4 \pm 4j$ (assuming you can measure the state directly). Your gains should depend on ω_0.

   (b) Design an estimator (observer) that reconstructs the state of the pendulum given measurements of $\dot{\theta}$. Pick the estimator roots to be at $s = -10 \pm 10j$. Provide an explicit, state space differential equation for your estimator design (again depending on ω_0).

   (c) Setting ω_0=5 rad/sec, plot the frequency response for the loop transfer function for the combined estimator, controller and process, and determine the gain and phase margin for the system.

2. (Friedland 2.1, 3.6, 7.2) Consider the inverted pendulum on a cart driven by an electric motor. The linearized equations of motion for the system are given by

$$\begin{aligned} \ddot{x} + \frac{k^2}{Mr^2R}\dot{x} + \frac{mg}{M}\theta &= \frac{k}{MRr}u \\ \ddot{\theta} - \left(\frac{M+m}{Ml}\right)g\theta - \frac{k^2}{Mr^2Rl}\dot{x} &= -\frac{k}{MRrl}u \end{aligned}$$

   where k is the motor torque constant, R is the motor resistance, r is the ratio of the linear forces applied to the cart ($\tau = rf$), and u is the voltage applied to the motor. The following numerical data may be used:

$$m = 0.1 \text{ kg} \qquad M = 1.0 \text{ kg} \qquad l = 1.0 \text{ m} \qquad g = 9.8 \text{ m/s}^2$$
$$k = 1 \text{ V} \cdot \text{s} \qquad R = 100\ \Omega \qquad r = 0.02 \text{ m}$$

   An observer for the inverted pendulum on a motor-driven cart is to be designed using the measurement of the displacement of the cart ($y = x$).

   (a) Find the matrices A, B, C, and D of the state-space characterization of the system.

   (b) Determine the observer gain for which the observer poles lie in a fourth-order Butterworth pattern of radius 5, i.e., the characterstic equation is to be

$$\left(\frac{s}{5}\right)^4 + 2.613\left(\frac{s}{5}\right)^3 + (2+\sqrt{2})\left(\frac{s}{5}\right)^2 + 2.613\left(\frac{s}{5}\right)^1 + 1 = 0.$$

(c) Plot the response of the observer states to an initial state estimate error of $(1, 1, 1, 1)$.

3. (Åström and Murray, Problem 7.2) Consider a system under a coordinate transformation $z = Tx$, where $T \in \mathbb{R}^{n \times n}$ is an invertible matrix. Show that the observability matrix for the transformed system is given by $\tilde{W}_o = W_o T^{-1}$ and hence observability is independent of the choice of coordinates.

4. (Åström and Murray, Problem 7.3) Show that if a system is observable, then there exists a change of coordinates $z = Tx$ that puts the transformed system into observable canonical form.