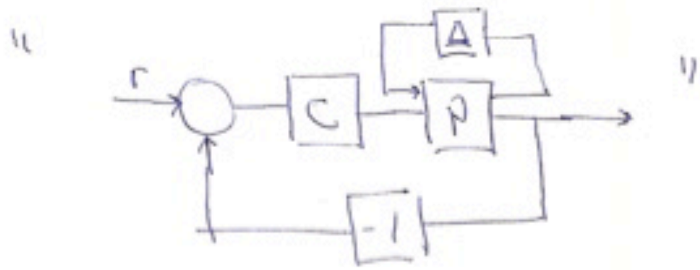
RMM 1 Mar 07

$\boxed{\text{Lecture 9-2: Robust Stability}}$



Norms of Signals & Systems

<u>Defn</u> Let V be a vector space over $\mathbb{R}$. A mapping $\|\cdot\|: V \to \mathbb{R}$ is a <u>norm</u> if it satisfies

1. $\|x\| \geq 0$ for all $x \in V$
2. $\|x\| = 0$ if and only if $x = 0$
3. $\|\alpha x\| = |\alpha| \cdot \|x\|$ for all $x \in V$, $\alpha \in \mathbb{R}$
4. $\|x + y\| \leq \|x\| + \|y\|$ (triangle inequality)

Examples of norms:

Name	$V = \mathbb{R}^n$	$V = \{u : (-\infty, \infty) \to \mathbb{R}\}$	$V, \|\cdot\|$
1-norm	$\|x\|_1 = \sum \|x_i\|$	$\|u\|_1 = \int_{-\infty}^{\infty} \|u(t)\| dt$	L_1
2-norm	$\|x\|_2 = \sqrt{\sum x_i^2}$	$\|u\|_2 = \left(\int_{-\infty}^{\infty} \|u(t)\|^2 dt\right)^{1/2}$	L_2
p-norm	$\|x\|_p = \left(\sum \|x_i\|^p\right)^{1/p}$	$\|u\|_p = \left(\int_{-\infty}^{\infty} \|u(t)\|^p dt\right)^{1/p}$	L_p
∞-norm	$\|x\|_\infty = \max \|x_i\|$	$\|u\|_\infty = \sup_t \|u(t)\|$	L_∞

<u>Defn</u> Let V be a vector space w/ norm $\|\cdot\|_a$ and W be a vector space with norm $\|\cdot\|_b$. The <u>induced norm</u> $\|\cdot\|_{a,b}$ of ~~[illegible]~~ a linear map $A: V \to W$ is given by

$$\|A\|_{a,b} = \sup_{\|v\|_a \le 1} \|Av\|_b$$

$$u \to \boxed{P} \to y \qquad y(t) = \overset{\text{ignore}}{\cancel{Ce^{At}x(0)}} + \underbrace{\int_0^{\infty} Ce^{A(t-\tau)} B u(\tau)\, d\tau}_{\text{linear map}}$$

<u>Thm</u> Let P be a stable system with representation $(A,B,C,0)$ and transfer function $\hat{A}(s) = C(sI-A)^{-1}B$. Then

$$\|P\|_{2,2} = \sup_{\omega} |\hat{A}(j\omega)| =: \|\hat{A}\|_\infty$$

↑ induced norm of the I/O <u>system</u>

↑ ∞-norm of the <u>transfer fcn</u>

<u>Pf</u> Let $\hat{y}(s) = \int_{-\infty}^{\infty} y(t) e^{st}\, dt$ (Laplace transform)

$$\|y\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{y}(j\omega)|^2\, d\omega \quad \text{(Parseval's thm)}$$

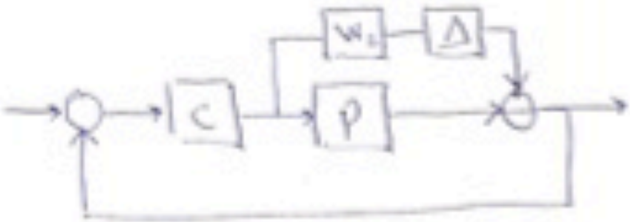
$$\hat{y}(s) = H(\overset{s}{j\omega})\, \hat{u}(s)$$

$$\Rightarrow \|y\|_2^2 = \|\hat{y}\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{A}(j\omega)|^2 |y(j\omega)|^2\, d\omega$$

$$\le \|\hat{A}\|_\infty^2 \cdot \frac{1}{2\pi}\int_{-\infty}^{\infty} |y(j\omega)|^2\, d\omega = \|\hat{A}\|_\infty^2 \|u\|_2^2$$

To complete, show $\exists$ input u such that $y(t)$ has gain $\|P\|_\infty$

$$\Rightarrow \max_{\|u\|_2 \le 1} \|y\|_2 = \max_{\|u\|_2 \le 1} \|Hu\|_2 = \|H\|_\infty$$



Additive uncertainty
$\|\Delta\|_\infty \le 1$ W_2 stable

Thm Assume the nominal system is stable. Then C provides robust stability $\Leftrightarrow$ $\|W_2 CS\|_\infty < 1$

Pf ($\Leftarrow$) Nominal stability $\Rightarrow$ L doesn't pass thru -1 and has n encirclements. $\tilde{L} = \tilde{P}C = PC + W_2 \Delta C$. Need to show $|1+\tilde{L}| > 0$ for all Δ ($\Rightarrow$ no new encirclements)

$$1+\tilde{L} = 1 + L + \Delta W_2 C = (1+L)\left(1+\frac{W_2 \Delta C}{1+L}\right) = (1+L)(1+W_2 \Delta S)$$

Since $\|W_2 CS\|_\infty < 1$ and $\|\Delta\|_\infty \le 1 \Rightarrow 1+W_2\Delta S$ never goes thru zero $\Rightarrow$ no change in net encirclements.

($\Rightarrow$) By contradiction. Assume perturbed system is stable but $\|W_2 CS\|_\infty \ge 1 \Rightarrow \exists\ \omega_0$ such that $|W_2 CS(j\omega_0)| = 1$.
$\Rightarrow W_2 CS(j\omega_0) = e^{j\phi_0}$ for some ϕ_0.
Let $\Delta = e^{j(\pi-\phi_0)}$. Then $\Delta W_2 CS(j\omega_0) = -1$
$\Rightarrow 1+\tilde{L} = (1+L)(1+\Delta W_2 CS(j\omega_0)) = 0 \Rightarrow$ not stable $\rightarrow\leftarrow$ //

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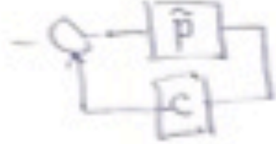
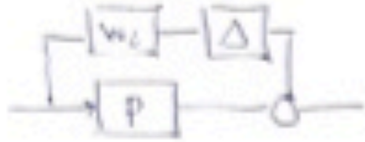
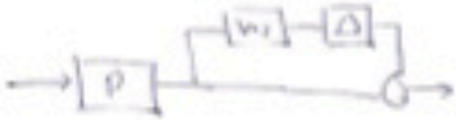
Robust Stability Conditions

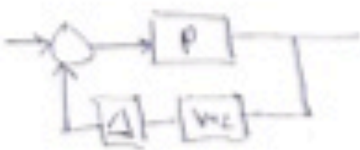




$\hat{P} = P + \Delta W_2$ $\qquad$ $\|W_2 C S\|_\infty < 1$



$\hat{P} = P(1 + \Delta W_2)$ $\qquad$ $\|W_2 T\|_\infty < 1$



$\hat{P} = \dfrac{P}{(1 + \Delta W_2 P)}$ $\qquad$ $\|W_2 P S\|_\infty < 1$

Remarks

1. Can derive all conditions via small gain theorem (by inspection)
2. Which perturbation you choose depends on how good the results are. Try to keep W_2 small
3. These conditions can all be "inverted" to see how much uncertainty you can tolerate

Additive: $\|W_2 C S\|_\infty = \|\frac{W_2}{P} T\|_\infty \le \|\frac{W_2}{P}\| \cdot |T|$

$\Rightarrow$ If $|\frac{W_2}{P}| < \frac{1}{|T|}$ $\forall \omega$ then robustly stable

$W_2 =$ "ΔP" (same condition as Wed)

Example cruise control

$$P(s) = \frac{b}{s+a} \qquad a = \frac{u_e \alpha_n^2 T'(\alpha_n v_e) - \rho C_d A v_e}{m} \approx 0.01 \pm 10\%$$

$$b = \frac{\alpha_n T(\alpha_n v_e)}{m} \approx 1.3 \pm 10\%$$

$$C(s) = k_p + \frac{k_i}{s+\beta} \qquad k_p = 0.5 \quad k_i = 0.1 \quad \beta = 0.1$$

Uncertainty #1: gain (b)

$$\tilde{P} = \frac{b + \delta b}{s+a} \qquad \text{Additive: } \tilde{P} = \frac{b}{s+a} + \frac{\overset{0.1b}{\cancel{\delta b}}}{s+a}\Delta \qquad W_2 = \frac{0.1b}{s+a}$$

$$\text{Multiplicative: } \tilde{P} = \frac{b}{s+a}(1 + 0.1\Delta) \qquad W_2 = 0.1$$

Use multiplicative: $\|W_2 T\|_\infty < 1$

Uncertainty #2: lag (a) $\quad a = 0.01 \pm 10\%$

$$\tilde{P} = \frac{b}{s+(a+\delta a)} \qquad \text{Additive: } \tilde{P} = P + W_2\Delta$$

$$W_2\Delta = \tilde{P} - P = \frac{b}{s+\tilde{a}} - \frac{b}{s+a} = \frac{b(a-\tilde{a})}{(s+\tilde{a})(s+a)}$$

$$\le \frac{b(0.1a\Delta)}{(s+a)(s+0.9a)} = \frac{0.1ba}{(s+a)(s+0.9a)}\Delta$$

$$\text{Multiplicative: } \tilde{P} = P(1 + W_2\Delta)$$

$$W_2\Delta = \frac{\tilde{P}}{P} - 1 = \frac{b/s+\tilde{a}}{b/s+a} - 1$$

$$= \frac{b(s+a)}{b(s+\tilde{a})} - \frac{s+\tilde{a}}{s+\tilde{a}} = \frac{a-\tilde{a}}{s+\tilde{a}}$$

$$\le \frac{0.1a}{s+0.9a}\Delta$$