CDS 110b: Lecture 9-1
Robust Stability

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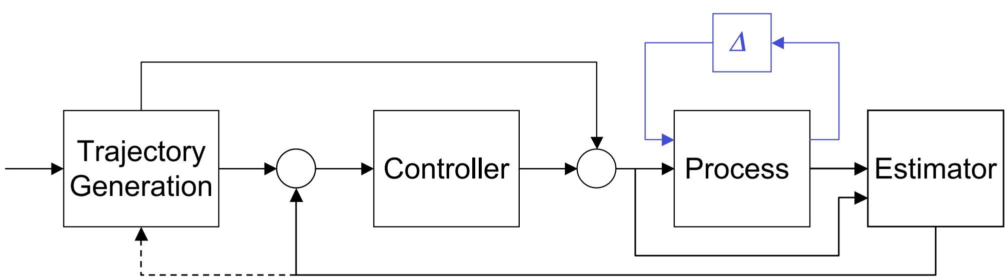
Goals:
- Describe methods for representing unmodeled dynamics
- Derive conditions for robust stability

Reading:
- AM06, Section 9.5, 12.1, 12.2 (norms + robust stability)
- Advanced: Doyle, Francis and Tannenbaum, Sections 4.1-4.3

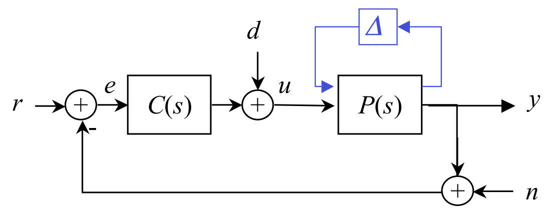
Game Plan: Robust Performance



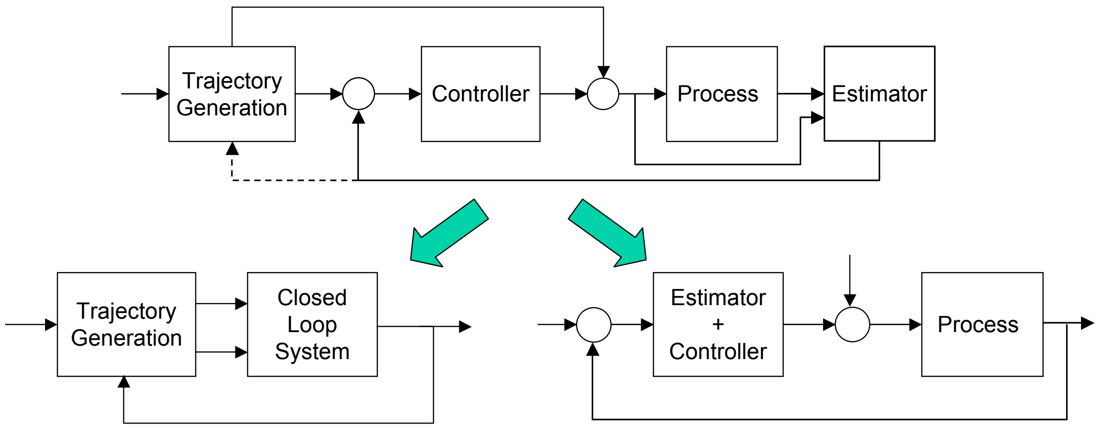
Robust Control
- Stability: bounded inputs → bounded outputs
- Performance: keep errors small

Robustness: do these things for all $\|\Delta\|_\infty < 1$

Simplifying case: focus on basic control loop (inner or outer loop)
- Stability = "internal stability"
- Performance = bounds on S & T
- Robustness = small gain theorem



Inner/Outer Loop Analysis



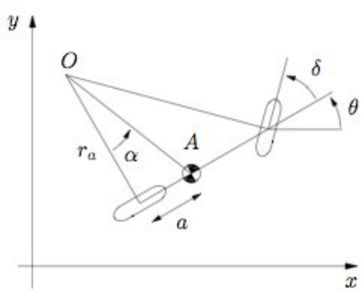
Outer loop: assume tracking is fast

- Replace tracking controller with closed loop dynamics
- Assumes tracking is much faster than trajectory generation
- Remaining uncertainty from system objective (ref) + environmental change

Inner loop: assume reference is slow

- Treat reference as constant input
- Design controller to provide sufficient bandwidth, tracking, disturbance rej.
- Reduces to standard (CDS 110a) control design; bandwidth requirements based on outer loop

Motivating Example: Vehicle Steering (AM06)



$$\dot{\xi} = v\cos(\alpha + \theta)$$
$$\dot{\eta} = v\sin(\alpha + \theta)$$
$$\dot{\theta} = \frac{v_0}{b}\tan\delta,$$

$$\frac{dw}{d\tau} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} w + \begin{bmatrix} \alpha \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} w$$

- Control the lateral dynamics to follow trajectory
- Normalize dynamics to simply functional form (AM06)

Control design

$$u = -k_1 w_1 - k_2 w_2 + k_r r,$$

- Choose gains so that closed loop dynamics have characteristic polynomial

$$p(s) = s^2 + 2\zeta_c\omega_c s + \omega_c^2.$$

$$\omega_c = 1,\ \zeta_c = 0.707$$

$$\Downarrow$$

$$k_1 = 100 \text{ and } k_2 = -35.86$$

Estimator design

$$\frac{d\hat{w}}{dt} = A\hat{w} + Bu + L(y - C\hat{w})$$

- Choose estimator gains to give closed loop observer with characteristic poly

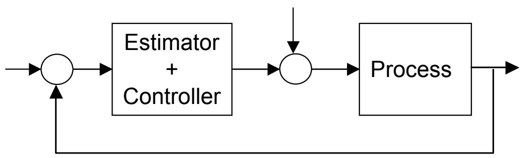
$$q(s) = s^2 + 2\zeta_o\omega_o s + \omega_o^2,$$

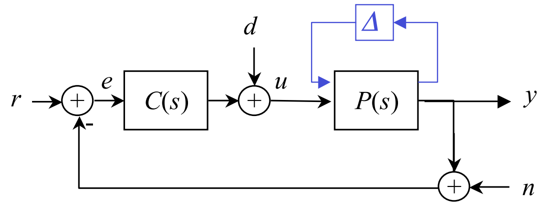
$$\omega_o = 20 \text{ and } \zeta_o = 2$$

$$\Downarrow$$

$$l_1 = 28.28 \text{ and } l_2 = 400$$

Closed Loop Steering Dynamics

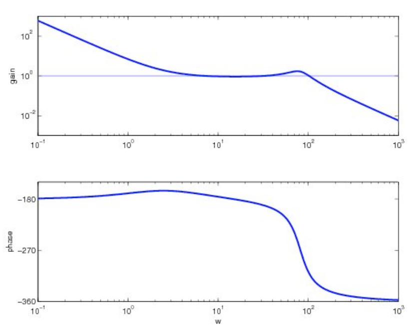




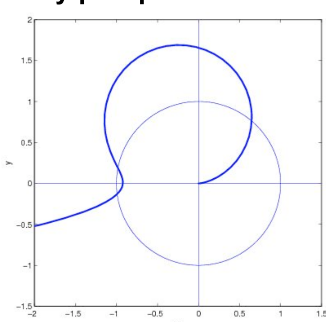
Process and controller dynamics

$$P(s) = \frac{0.5s + 1}{s^2}, \qquad C(s) = \frac{-11516s + 40000}{s^2 + 42.4s + 6657.9}.$$

Bode Plot



Nyquist plot



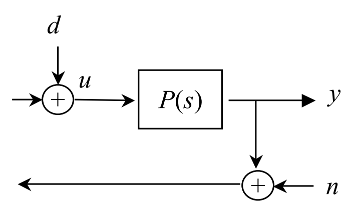
Closed loop not robust!

- No net encirclements, but phase margin is very small (around 7 deg)
- Small change in process dynamics will cause system to go unstable
- Not clear from design what went wrong - closed loop dynamics look reasonable
- Problem is not limited to eigenvalue placement (HW: show for LQR + KF)

Modeling Uncertainty

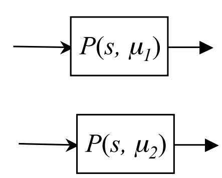
Noise and disturbances

- Model the amount of noise by its signal strength in different frequency bands
- Can model signal strength by peak amplitude, average energy, and other norms
- Typical example: Dryden gust models (filtered white noise)



Parametric uncertainty

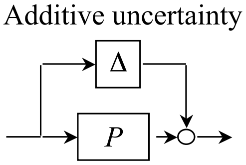
- Unknown parameters or parameters that vary between plants
- Typically specified as tolerances on the nominal parameters



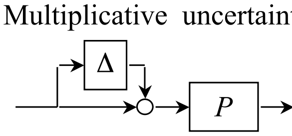
Unmodeled *dynamics*

- High frequency dynamics can be excited by control loops
- Use bounded operators to account for unmodeled dynamics:

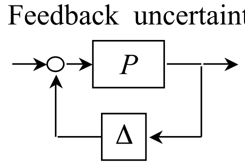
Additive uncertainty



Multiplicative uncertainty

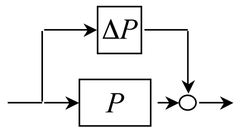


Feedback uncertainty



Robust Stability via Nyquist

Additive uncertainty case

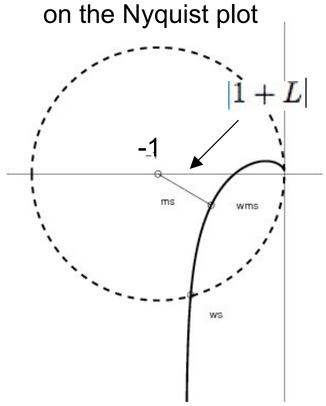


$$\tilde{P} = P + \Delta P$$

$$\tilde{L} = \tilde{P}C = PC + \Delta PC$$

How much ΔP can we tolerate?

- Look at distance from the critical point on the Nyquist plot



$$|C\Delta P| < |1 + L|$$

$$\Downarrow$$

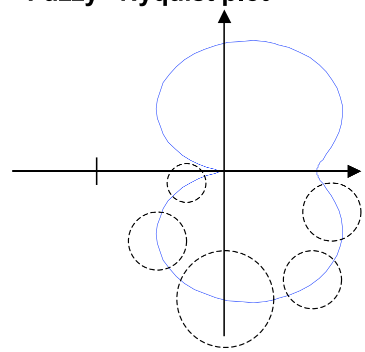
$$|\Delta P| < \left|\frac{1+PC}{C}\right|$$

$$\Downarrow$$

$$\left|\frac{\Delta P}{P}\right| < \frac{1}{|T|}$$

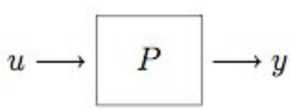
(must hold at all frequencies)

"Fuzzy" Nyquist plot



- Can handle any uncertainty within "tube" around nominal process (no new encirc'ts)
- Caution: requires perturb-ations to be *stable*
- Conservative condition: allows *any* variation w/in tube; reality may be more kind

System Norms



$$y(t) = \underbrace{Ce^{At}x(0)}_{\text{Assume zero}} + \underbrace{\int_0^t Ce^{A(t-\tau)}Bu(\tau)\,d\tau}_{\text{Linear map}}$$

What is the "gain" from *u* to *y*?

- Would like a single number, not something dependent on frequency
- Answer depends on what you choose as the norms for the input and ouputs

2-norm of a signal

- Define like the two norm of a vector (but with integral instead of a sum):

$$\|u\|_2 = \left(\int_{-\infty}^{\infty} u^2(t)dt\right)^{1/2}$$

- Corresponds to the "energy" contained in a signal
- Caution: not defined for sinusoids (!)

Induced norm of a system

- Look at maximum norm of the output given all possible unit-norm inputs

	$\|u\|_2 = \left(\int_{-\infty}^{\infty} u^2(t)dt\right)^{1/2}$	$\|u\|_\infty = \sup_t \|u(t)\|$
$\|y\|_2$	$\sup_\omega \|H(j\omega)\|$	∞
$\|y\|_\infty$	$\frac{1}{2\pi}\left(\int_{-\infty}^{\infty} \|H(j\omega)\|^2 d\omega\right)^{1/2}$	$\int_{-\infty}^{\infty} \|Ce^{At}B\| dt$

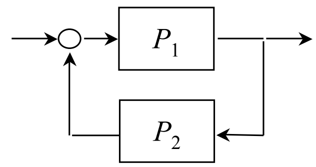
- Common choice for control purposes is the "induced 2-norm"

$$\|H\|_\infty = \sup_\omega |H(i\omega)|.$$

also called the "infinity norm" (of a transfer function)

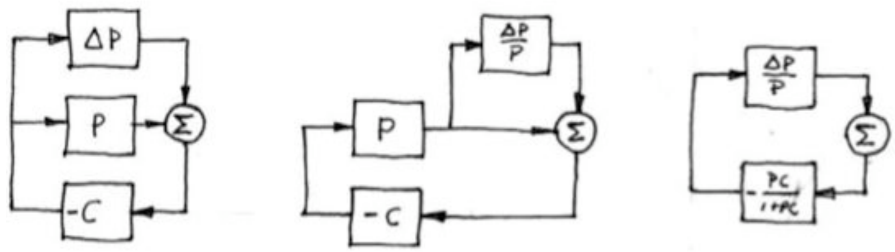
Small Gain Theorem

Theorem (Small gain theorem) Consider two stable, linear time invariant processes with transfer functions $P_1(s)$ and $P_2(s)$. The feed-back interconnection of these two systems is stable if $\| P_1 P_2 \|_\infty < 1$.



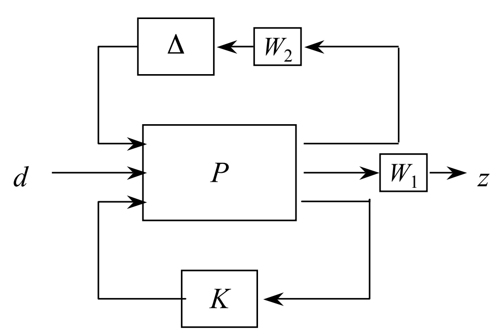
Application to robust stability

- Use block diagram algebra to convert system into form of small gain theorem



$$\left\| \frac{\Delta P}{P} \cdot T \right\|_\infty < 1 \quad \Longrightarrow \quad \left| \frac{\Delta P}{P} \right| < \frac{1}{|T|} \quad \forall \omega$$

Preview: Robust *Performance*



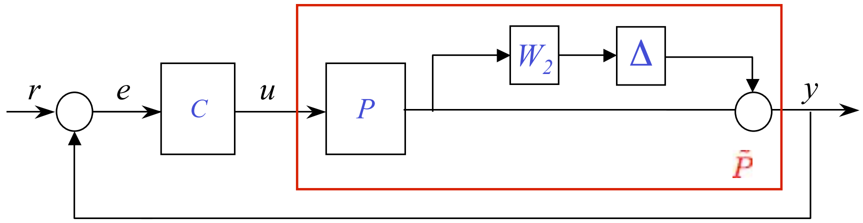
d	disturbance signal
z	output signal
Δ	uncertainty block
W_2	performance weight
W_2	uncertainty weight

Goal: guaranteed performance in presence of uncertainty

$$\|z\|_2 \leq \gamma \, \|d\|_2 \quad \text{for all} \quad \|\Delta\| \leq 1$$

- Compare energy in disturbances to energy in outputs
- Use frequency weights to change performance/uncertainty descriptions
- “Can I get X level of performance even with Y level of uncertainty?”

Robust Stability Using Frequency Domain Weighting



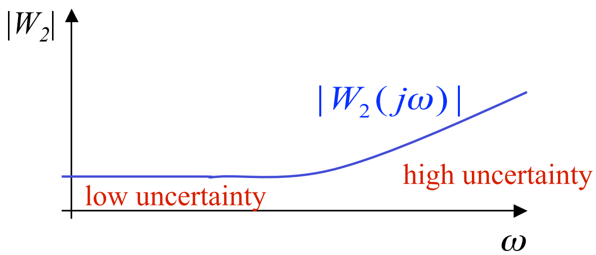
Multiplicative Uncertainty

- Model plant as nominal with additional dynamics given by $W_2\,\Delta$

$$\tilde{P} = P(1 + W_2\Delta)$$

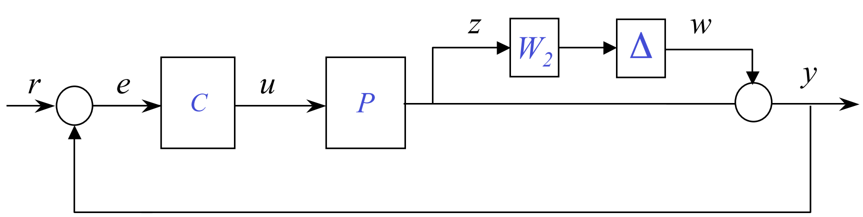
W_2 = frequency weight

Δ = uncertainty; require $|\Delta(jw)| \leq 1$



- Δ allows *any* dynamics to be inserted into the plant
- Can be used to model parameter uncertainty or unmodeled dynamics

Complementary Sensitivity and Robustness



Thm *A controller C provides robust stability to multiplicative perturbations if and only if*

$$\boxed{|W_2(j\omega)\, T(j\omega)| < 1} \text{ for all } \omega.$$

where

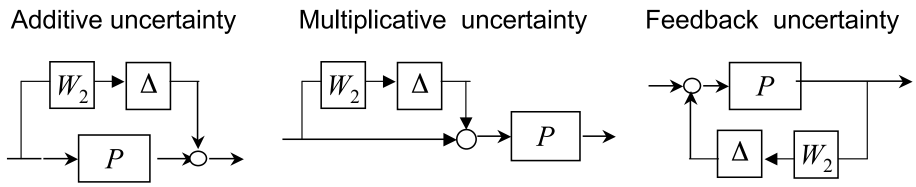
Complementary sensitivity function

$$T := \frac{PC}{1 + PC} = H_{zw}$$

Intuition: H_{zw} represents the transfer function seen by the weighted uncertainty $W_2\,\Delta$

Note: this theorem guarantees stability for any *transfer function* $\Delta(s)$ satisfying $|\Delta(j\omega)| < 1 \Rightarrow$ allows unmodeled *dynamics* (as well as parametric uncertainty)

Models for Uncertainty



Each model describes a class of process dynamics:

- Additive: $\tilde{P} = P + W_2\Delta$
- Multiplicative: $\tilde{P} = P(1 + W_2\Delta)$
- Feedback: $\tilde{P} = P/(1 + PW_2\Delta)$

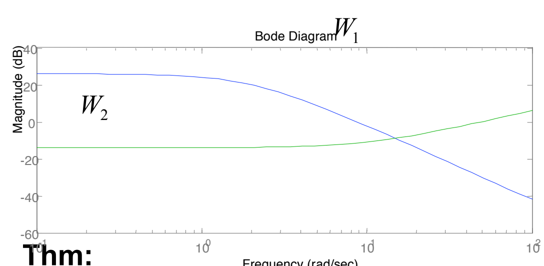
Use $||W_2||$ to shape the unmodeled dynamics; $||\Delta||_\infty < 1$ in all cases

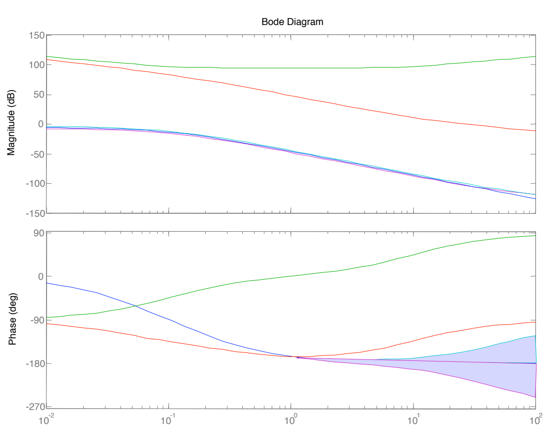
Robust stability conditions given by small gain theorem

- Compute transfer function around Δ block and require that this be < 1
- (If not, can choose Δ with $||\Delta||_\infty \leq 1$ to destabilize)

Example: Robust Cruise Control

$$P(s) = \frac{1/m}{s + b/m} \cdot \frac{r}{s+a}\left(1 + W_2\Delta\right)$$





Thm:

- Performance: $\left| W_1 \frac{1}{1+L} \right| < 1 \quad \forall \omega$
- Robust Stability: $\left| W_2 \frac{L}{1+L} \right| < 1 \quad \forall \omega$