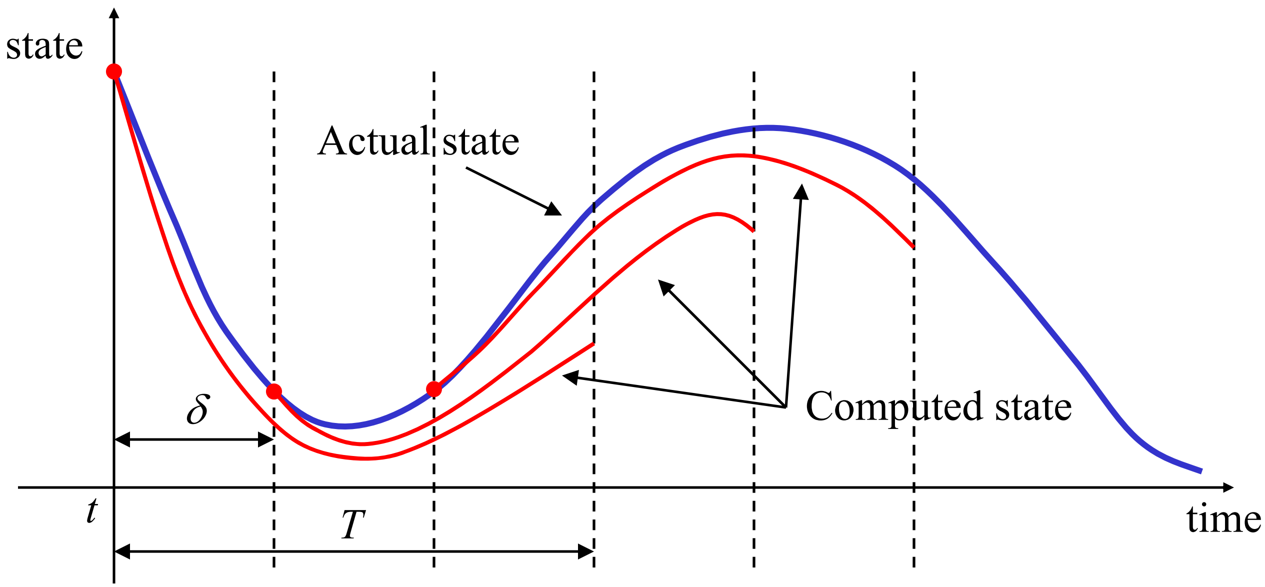
# Continuous time, unconstrained nonlinear RHC

**Solve finite time optimization over *T* seconds and implement first $\delta$ seconds**

$$J_T^*\left(x(t),u(\cdot)\right)=\min_{u(\cdot)}\int_t^{t+T} q(x(\tau),u(\tau))\,d\tau+V(x(t+T))$$

$$\dot{x}=f(x,u)$$

Incremental cost — Terminal cost




Murray, Hauser et al
SEC chapter (IEEE, 2002)

**Requires that computation time be small relative to time horizons**

- Initial implementation in process control, where time scales are fairly slow.
- Real-time trajectory generation enables implementation on faster systems.

# Stability of Receding Horizon Control

**RHC can destabilize systems if not done properly**

- For properly chosen cost functions, get stability with $T$ sufficiently large.
- For shorter horizons, $V$ has to be chosen properly to avoid instability.

**The choice of the terminal cost**

- Best choice would be $V(x) = J_\infty^*(x)$ such that the optimal finite and infinite costs are the same. (Not possible, if the optimal value function were available there would be no need to solve a trajectory optimization problem.)
- The terminal cost must account for the discarded tail by ensuring that the origin can be reached from the terminal state $x(t+T)$ in an efficient manner as measured by $q$.

  One way to do this is to use an appropriate control Lyapunov function (CLF).

# Control Lyapunov Function (CLF)

**Definition**

A control Lyapunov function (CLF) is a $C^1$, proper, positive definite function $V : R^n \to R_+$ such that

$$\inf_u \left[ \dot{V}(x,u) \right] \leq 0$$

where

$$\dot{V}(x,u) = \frac{dV}{dx}\frac{dx}{dt} = \frac{dV}{dx} f(x,u)$$

denotes the directional derivative in direction $f(x, u)$.

**Meaning**

If it is possible to make the derivative negative at every point by an appropriate choice of $u$ then we can stabilize the system with $V$ as a Lyapunov function for the closed loop.

It can be shown that the existence of a CLF is equivalent to the existence of an asymptotically stabilizing control law u = k(x).

# Stability of Receding Horizon Control

**Theorem (Jadbabaie & Hauser, 2002)**

Suppose that the terminal cost $V(x)$ is a control Lyapunov function such that

$$\min_u \left(\dot{V} + q\right)\left(x, u\right) \leq 0$$

for each $x$ in $\Omega_r = \{x : V(x) < r^2\}$, for some $r > 0$. Then, for every $T > 0$ and $\delta$ in $(0, T\,]$, the resulting receding horizon trajectories go to zero exponentially fast.

**Remarks**

- Earlier approach used terminal trajectory constraints, hard to implement in real-time.
- CLF terminal cost is difficult to find in general, but LQR-based solution at equilibrium point often works well - choose $V = x^T P\, x$ where $P$ = Riccati solution.

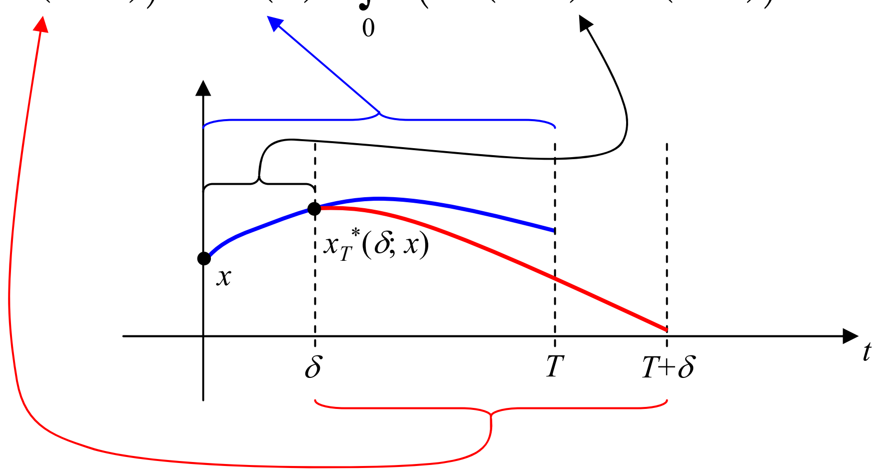
# Main ingredient of the proof

Denote with $x^u(\tau; x)$ the state trajectory at time $\tau$ starting from initial state $x$ and applying a control trajectory $u(\cdot)$.

Let $(x_T^*, u_T^*)(\cdot\,; x)$ denote an optimal trajectory of the finite horizon optimal control problem with horizon $T$.

Assume $x_T^*(T; x) \in \Omega_r = \{x : V(x) < r^2\}$, for some $r > 0$. Then, for each $\delta \in [0, T]$, the optimal cost from $x_T^*(\delta; x)$ satisfies

$$J_T^*\left(x_T^*\left(\delta; x\right)\right) \le J_T^*\left(x\right) - \int_0^\delta q\left(x_T^*\left(\tau; x\right), u_T^*\left(\tau; x\right)\right) d\tau$$



# Proof sketch

Let $\left(\tilde{x}(t), \tilde{u}(t)\right)$ , $t \in [0, 2T]$, be the trajectory obtained by concatenating $(x_T^{\,*}, u_T^{\,*})(t\,; x)$, $t \in [0, T]$, and $(x^k, u^k)(t - T\,; x_T^{\,*}(T\,; x))$, $t \in [T, 2T]$, which are closed-loop trajectories corresponding to a feedback law $u = k(x)$ such that $\left(\dot{V} + q\right)\left(x, k(x)\right) \le 0$.

Consider the cost of using $\tilde{u}(\cdot)$ for $T$ seconds at the initial state $x_T^{\,*}(\delta\,; x))$, $\delta \in [0, T]$

$$
\begin{aligned}
J_T\left(x_T^*(\delta; x), \tilde{u}(\cdot)\right) &= \int_{\delta}^{T+\delta} q\left(\tilde{x}(\tau), \tilde{u}(\tau)\right) d\tau + V\left(\tilde{x}(T+\delta)\right) \\
&= J_T^*(x) - \int_0^{\delta} q\left(x_T^*(\tau; x), u_T^*(\tau; x)\right) d\tau - V\left(x_T^*(T; x)\right) \\
&\quad + \int_T^{T+\delta} q\left(\tilde{x}(\tau), \tilde{u}(\tau)\right) d\tau + V\left(\tilde{x}(T+\delta)\right) \\
&\le J_T^*(x) - \int_0^{\delta} q\left(x_T^*(\tau; x), u_T^*(\tau; x)\right) d\tau
\end{aligned}
$$

where we have used the facts that $q\left(\tilde{x}(\tau), \tilde{u}(\tau)\right) \le -\dot{V}\left(\tilde{x}(\tau), \tilde{u}(\tau)\right)$ for all $\tau \in [T, 2T]$

and due to optimality $J_T^*\left(x_T^*(\delta; x)\right) \le J_T\left(x_T^*(\delta; x), \tilde{u}(\cdot)\right)$