Lecture 6.2 - Differential Flatness

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Trajectory generation: find $x_d(t)$, $u_d(t)$ that satisfy

$$(*) \qquad \dot{x}_d = f(x_d, u_d) \qquad x_d(0) = x_0 \quad x_d(T) = x_f$$

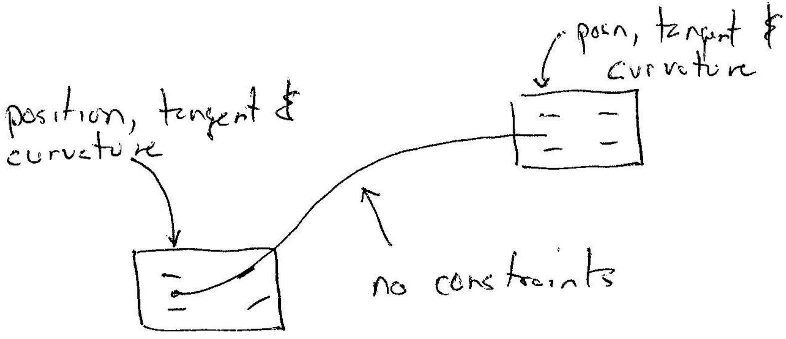
In addition, we may wish to satisfy additional constraints

- Input saturation: $|u(t)| < M$
- State constraints: $g(x) \leq 0$
- Tracking: $h(x) = r(t)$
- Optimization: $\min \int_0^T L(x,u)\,dt + V(x(T), u(T))$

  Special case: $L(x,u) = \| h(x) - r(t) \|^2 \Rightarrow$ tracking

In principle, we can use maximum principle to solve this, but often difficult to work out <u>explicit</u> solutions for x_d, u_d from two point boundary value problem $\Rightarrow$ look for other solutions.

Motivating example: kinematic car



$$\dot{x} = \cos\theta\, v$$
$$\dot{y} = \sin\theta\, v$$
$$\dot{\theta} = \frac{v}{\ell} \tan\phi$$

Given x, y, we can solve for $\theta = \operatorname{atan}(\dot{y}/\dot{x})$, $v = \dot{x}/\cos\theta$, $\phi = \operatorname{atan}\left(\frac{\ell}{v\dot{\theta}}\right) \Rightarrow$ get trajectory for full state. Initial &

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Differential Flatness

Defn A nonlinear system (*) is differentially flat if there exists a function α such that

$$z = \alpha(x, u, \dot{u}, \dots, u^{(q)})$$

and we can write the solutions of the differential equation as functions of z and a finite number of derivatives

$$x = \beta(z, \dot{z}, \dots, z^{(p)})$$
$$u = \gamma(z, \dot{z}, \dots, z^{(p)})$$

Example: for kinematic car $z = \alpha(\vec{x}) = (x, y)$

Remarks

1. For a differentially flat system, the flat outputs, z, completely define the feasible trajectories of the system
2. The number of flat outputs = number of system inputs
3. General theory for determing if a system is flat is hard; usually guess & check
4. General classes of systems that are flat:
   - Reachable linear systems
   - Mechanical systems with m configuration variables and m inputs
   - Feedback linearizable ~~system~~ nonlinear systems

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Using flatness to plan trajectories

Suppose we wish to generate a feasible trajectory for NL system

$$\dot{x} = f(x, u) \qquad x(0) = x_0 \qquad x(T) = x_f$$

If system is differential flat then

$$x(0) = \beta\left(z(0), \dot{z}(0), \ldots, z^{(p)}(0)\right) = x_0$$
$$x(T) = \gamma\left(z(T), \dot{z}(T), \ldots, z^{(p)}(T)\right) = x_f$$

Find any $z(0), \dot{z}(0), \ldots$ & $z(T), \dot{z}(T), \ldots$ that satisfy these constraints (often unique). Choose

$$z(t) = \sum_{i=1}^{N} \alpha_i \psi_i(t)$$

$\alpha_i \in \mathbb{R}$
ψ^i smooth basis fcns

$$\dot{z}(t) = \sum_{i=1}^{N} \alpha_i \dot{\psi}_i(t)$$

$\vdots$

$$z^{(p)}(t) = \sum_{i=1}^{N} \alpha_i \psi_i^{(p)}(t)$$

$$\begin{bmatrix} \psi_1(0) & \psi_2(0) & \cdots & \psi_N(0) \\ \dot{\psi}_1(0) & \dot{\psi}_2(0) & \cdots & \dot{\psi}_N(0) \\ \vdots & & & \\ \psi_1^{(p)}(0) & \psi_2^{(p)}(0) & \cdots & \psi_N^{(p)}(0) \\ \psi_1(T) & & \cdots & \psi_N(T) \\ \vdots & & & \vdots \\ \psi_1^{(p)}(T) & & & \psi_N^{(p)}(T) \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_N \end{bmatrix} = \underbrace{\begin{bmatrix} z(0) \\ \dot{z}(0) \\ \vdots \\ z^{(p)}(0) \\ \hline z(T) \\ \vdots \\ z^{(p)}(T) \end{bmatrix}}_{b}$$

Example Nonholonomic integrator

$$\dot{x}_1 = u_1 \qquad \dot{x}_2 = u_2 \qquad \dot{x}_3 = x_2 u_1$$

Differentially flat w/ $z = (x_1, x_3)$

$$x_1 = z_1 \qquad x_2 = \dot{x}_3/\dot{x}_1 = \dot{z}_2/\dot{z}_1$$

$$x_3 = z_2 \qquad u_1 = \dot{z}_1$$

$$u_2 = \dot{x}_2 = \frac{d}{dt}\left(\dot{z}_2/\dot{z}_1\right) = \frac{\ddot{z}_2}{\dot{z}_1} - \frac{\dot{z}_2 \ddot{z}_1}{\dot{z}_1^2}$$

Note: intuition says taking derivatives is "noisy." Here we are taking derivatives of basis functions $\Rightarrow$ noise isn't an issue.

$$z_1(t) = \sum a_{1,i} \psi_{1,i}(t) \qquad z_2(t) = \sum a_{2,i} \psi_{2,i}(t)$$

Generate a trajectory from origin to a point $x = (x_{1f}, x_{2f}, x_{3f})$

Choose basis functions as simple polynomials

$$\psi_{1,1}(t) = 1 \qquad \psi_{1,2}(t) = t \qquad \psi_{13}(t) = t^2 \qquad \psi_{14}(t) = t^3$$

$$\psi_{2,1}(t) = 1 \qquad \psi_{2,2}(t) = t \qquad \psi_{2,3}(t) = t^2 \qquad \psi_{24}(t) = t^3$$

$$\begin{array}{c} z_1 \\ \dot{z}_1 \\ z_2 \\ \dot{z}_2 \\ \\ \\ \\ \\ \end{array} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & T & T^2 & T^3 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2T & 3T^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & T & T^2 & T^3 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2T & 3T^2 \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{12} \\ a_{13} \\ a_{21} \\ a_{22} \\ a_{23} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ x_{1f} \\ 1 \\ x_{3f} \\ x_{2f} \end{bmatrix}$$

8x8 matrix
Invertible for most T
Doesn't depend on x_0, x_f

Some freedom here

Remarks

1. There are several remaining degrees of freedom (T, 1, etc) that have to be specified $\Rightarrow$ solutions are not unique
2. C[illegible] parameterize and solve least squares

Using flatness for constrained, sub-optimal trajectory generation

Return to full problem

$\cancel{\dot{x} = Ax}$, $\min \int_0^T L(x,u)\,dt + V(x(T), u(T))$

subject to

$\dot{x} = f(x,u) \qquad g(x,u) \leq 0 \quad \leftarrow$ state + input constraints

If system is differentially flat, can write $\quad z = \sum a_i \psi_i(t)$

$$x = \beta(z, \dot{z}, \ldots, z^{(q)}) = \beta(a,t)$$
$$u = \gamma(\qquad\qquad) = \gamma(a,t)$$

Can rewrite entire problem in terms of α

$$\min \int_0^T L(\beta(\alpha,t), \gamma(a,t))\,dt + V(\beta(\alpha,T), \gamma(\alpha,T))$$

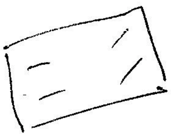
$$g(\beta(\alpha,t), \gamma(\alpha,t)) \leq 0 \qquad \text{No dynamics}$$

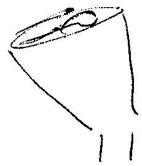
This is an optimization problem in fixed set of parameters: given a, can compute all quantities. Very good numerical tools available.

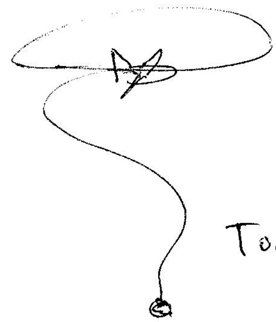
Remarks

1. $g(\beta(\alpha,t), \gamma(\alpha,t))$ give constraint for each $t \Rightarrow$ infinite # of constraints $\Rightarrow$ hard. Can replace with $g(\beta(a,t_i), \gamma(a,t_i))$ for $t_1, t_2, \ldots, t_M \Rightarrow$ finite approx.
2. Can extend to non-flat, etc (more next week)
3. Sub-optimal, but fast

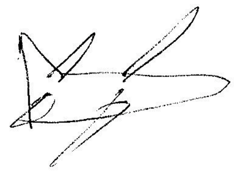
Examples of flat systems


Kin car


Ducted fan


Towed cable


N-trailer


Lift (elevator) $\ll$ Lift (wings)