CALIFORNIA INSTITUTE OF TECHNOLOGY
Control and Dynamical Systems

CDS 110b

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In this set of lectures we introduce the concept of the norm of a (linear) system and show how this can be used to specify stability and performance measures.

Reading:

- DFT, Chapters 2 and 3

1 Introduction

One of the key uses of feedback is to provide robustness to uncertainty. We have already seen several examples of this in the previous chapters, from tracking a reference signal whose evolution is not known ahead of time, to rejecting disturbances that might affect the process or our measurements of it. In this chapter we generalize the classes of uncertainty that we can analyze to include uncertainty in the process that we are trying to control.

Robustness to process variations is the ability of a closed loop system to be insensitive to component variations that make up the system. It is one of the most useful properties of feedback and is what make it possible to design feedback systems based on strongly simplified models.

One of the key issues in studying robustness is to describe variations in system dynamics. Using state space representations for a system, uncertainty can be captured by varying the parameters of a system (e.g., the elements of the A, B, C and D matrices). There are however many variations that are not captured by such an approach. For example, some dynamic phenomena may have been neglected during the modeling stage to keep the the size of the model tractable. These *unmodeled dynamics* cannot be accounted for by simply modifying the parameters of the simplified system. Similarly, there may also be small time delays that have been neglected in a model, in part because these are difficult to analyze using ODEs.

As an example of this type of uncertainty, consider the speed controller that has been discussed throughout CDS 110a. We have designed several different controllers for this system using a model that was derived in Chapter 2 of AM05. The original model considered the dynamics of the forward motion of the vehicle and the torque characteristics of the engine and transmission. It did not, however, include a detailed model of the engine dynamics (whose

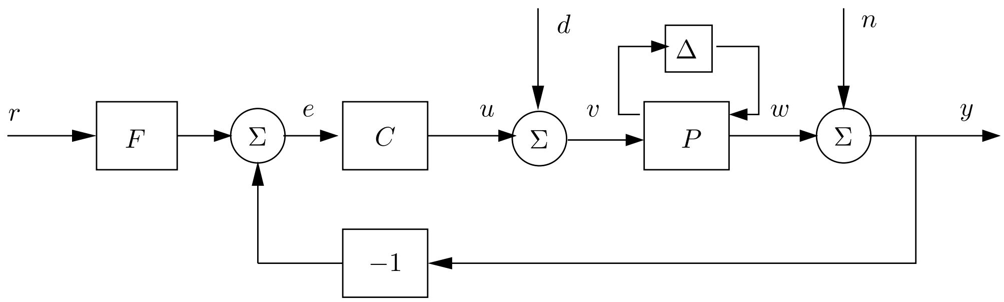


Figure 1: Block diagram of a basic feedback loop.

combustion processes are extremely complex), nor the slight delays that can occur in modern electronically controlled engines (due to the processing time of the embedded computers). Rather than try to include all of these details in our model, and how they change over time, we can instead try to show that the controllers that we have designed using our simplified models are robust with respect to the unmodeled dynamics. Under the assumption that the dynamics that we initially ignored are "small" compared with the dominant dynamics of the system, we expect that our controllers will work when implemented on the full system.

Transfer functions provide a natural way of modeling these types of uncertainty (and many others). The basic idea is to model the "unmodeled" dynamics by including a transfer function in the system desription whose frequency response is bounded, but otherwise unspecified. So, for example, we might model the engine dynamics in the speed control example as a system that very quickly provides the torque that is requested through the throttle, giving a small deviation from the simplified model, which assumed the torque response was instantaneous. This technique can also be used in many instances to model parameter variations, allowing a quite general approach to uncertainty management. One limitation of the tools we present here is that they are usually restricted to linear systems, although some nonlinear extensions have been developed.

A block diagram representation of this approach is shown below: In this figure, the block Δ represents the unmodeled dynamics of the system and we think of Δ as "small" (in a sense to be made precise later). Our goal is to design a controller C such that the closed loop performance of the system satisfies a given specification for *any* $\Delta \leq M$. We say that such a controller is *robust* with respect to the (bounded) perturbation Δ. In order to make these concepts precise, we define the *norms* of signals and systems.

2 Norms of Signals

Definition 1. A set V is a *linear (vector) space over* $\mathbb{R}$ if we can define operations of addition (of vectors) and multiplication (by scalars) such that for every $x, y \in V$ and $\alpha, \beta \in \mathbb{R}$ we have

$$\alpha x + \beta y \in V \qquad \alpha(x+y) = \alpha x + \alpha y \in V$$

and there exists a zero element $\vec{0} \in V$ such that $\vec{0} = x - x = 0 \cdot x$ (where 0 is the real number zero).

Examples:

1. $V = \mathbb{R}^n$
2. $\mathcal{C}^n[t_0, t_1]$ = the space of continous functions mapping the interval $[t_0, t_1]$ to $\mathbb{R}^n$.
3. $\mathcal{P}^n(-\infty, \infty)$ = the set of piecewise continous functions on $(-\infty, \infty)$ taking values in $\mathbb{R}^n$.

These last two examples are "infinite dimensional", in the sense that there is no finite basis for the space.

Definition 2. Let V be a vector space over $\mathbb{R}$. A mapping $\|\cdot\| : V \to \mathbb{R}^n$ is a *norm* if it satisfies:

1. $\|x\| \geq 0$ for all $x \in V$
2. $\|x\| = 0$ if and only if $x = 0$
3. $\|\alpha x\| = |\alpha| \cdot \|x\|$ for all $x \in V$ and $\alpha \in \mathbb{R}$
4. $\|x + y\| \leq \|x\| + \|y\|$ (triangle inequality)

The definitions are correct even if V is infinite dimensional. For example, if we take $\mathcal{C}(-\infty, \infty) = \{x : (-\infty, \infty) \to \mathbb{R}, \text{ continuous}\}$, then the zero element is the zero *function* and scaling and addition are done pointwise in time:

$$(u+v)(t) = u(t) + v(t) \qquad (\alpha u)(t) = \alpha u(t)$$

Examples of norms:

Name	$V = \mathbb{R}^n$	$V = \{u : (-\infty, \infty) \to \mathbb{R}\}$
1-norm	$\|x\|_1 = \sum_{i=1}^n \|x_i\|$	$\|u\|_1 = \int_{-\infty}^{\infty} \|u(t)\| \, dt$
2-norm	$\|x\|_2 = \sqrt{\sum_{i=1}^n \|x_i\|^2}$	$\|u\|_2 = \left(\int_{-\infty}^{\infty} \|u(t)\|^2 \, dt\right)^{1/2}$
p-norm	$\|x\|_p = \sqrt[p]{\sum_{i=1}^n \|x_i\|^p}$	$\|u\|_p = \left(\int_{-\infty}^{\infty} \|u(t)\|^p \, dt\right)^{1/p}$
∞-norm	$\|x\|_\infty = \max_i \|x_i\|$	$\|u\|_\infty = \sup_t \|u(t)\|$

The operation sup is the least upper bound of a function.

Example: 2 norm on $\mathcal{C}(-\infty,\infty)$

1. $\|u\|_2 = \left(\int_{-\infty}^{\infty} |u(t)|^2\,dt\right)^{1/2} > p$ ✓

2. $$\begin{aligned} \|u\|_2 = 0 \quad &\implies \quad \int_{-\infty}^{\infty} |u(t)|^2 = 0 \\ &\implies \quad u(t) = 0 \text{ on any interval} \\ &\implies \quad u(t) = 0 \text{ for all } t \text{ (by continuity)} \implies u = 0 \checkmark \end{aligned}$$

3. $\|\alpha u\|_2 = \left(\int_{-\infty}^{\infty} |\alpha u(t)|^2\,dt\right)^{1/2} = \alpha\|u\|_2$ ✓

4. Triangle inequality: $\|u+v\|_2 \le \|u\|_2 + \|v\|_2$

   Proof: Exercise. Use the fact that $|u(t)+v(t)| \le |u(t)| + |v(t)|$.

Definition 3. A *normed linear space* V is a linear (vector) space V equipped with a norm $\|\cdot\|$.

Examples:

$$\begin{array}{lll} L_1 & V = P(-\infty,\infty) & \|\cdot\|_1 \\ L_2 & V = P(-\infty,\infty) & \|\cdot\|_2 \\ L_\infty & V = P(-\infty,\infty) & \|\cdot\|_\infty \end{array}$$

3 Linear Mappings and Induced Norms

Let V and W be linear spaces over $\mathbb{R}$.
Definition 4. A mapping $A : V \to W$ is a *linear map* if

$$A(\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 A v_1 + \alpha_2 A v_2$$

for all $v_1, v_2 \in V$ and $\alpha_1, \alpha_2 \in \mathbb{R}$.

Examples:

1. The set of $n \times n$ matrices are a linear map from $\mathbb{R}^n$ to $\mathbb{R}^n$ using the usual rules of multiplying a vector by a matrix.

2. Let $V = \mathcal{P}[0,1]$ and $W = \mathbb{R}$. Then the operator given by

   $$Av = \int_0^1 v(t)\,dt$$

   is a linear mapping from V to W.

3. Let $V = \mathcal{P}[0, \infty]$ and $W = \mathcal{P}[0, \infty]$. Then the operator given by

$$w(t) = \int_0^t e^{a(t-\tau)} v(\tau)\, d\tau$$

is a linear mapping from V to W.

Definition 5. Let V be a vector space with norm $\|\cdot\|_a$ and W be a vector space with norm $\|\cdot\|_b$. Then the *induced norm* $\|\cdot\|_{a,b}$ of $A : V \to W$ is given by

$$\|A\|_{a,b} = \sup_{\|v\|_a \le 1} \|Av\|b$$

Example: matrix norms

$V = \mathbb{R}^n$	$W = \mathbb{R}^n$	$A : V \to W$
$\|v\|_2 = \sqrt{v^T v}$	$\|w\|_2 = \sqrt{w^T w}$	$\|A\|_{2,2} = \sqrt{\lambda_{\max}(A^T A)}$

4 Norms of Systems

$$u \longrightarrow \boxed{\;P\;} \longrightarrow y \qquad\qquad y(t) = \underbrace{Ce^{At}x(0)}_{\text{Assume zero}} + \underbrace{\int_0^t Ce^{A(t-\tau)} Bu(\tau)\, d\tau}_{\text{Linear map}}$$

Q: what is the *induced norm* of the system P, thought of as an input/output mapping of signals?

A: Depnds on the norm of the signals

Theorem 1. *Let Σ be a stable system with state space representation $(A, B, C, 0)$, transfer function $H(s) = C(sI - A)^{-1}B$. Then the following table gives the input/output norm of Σ*

	$\|u\|_2 = \left(\int_{-\infty}^{\infty} u^2(t)dt\right)^{1/2}$	$\|u\|_\infty = \sup_t \|u(t)\|$
$\|y\|_2$	$\sup_\omega \|H(j\omega)\|$	∞
$\|y\|_\infty$	$\frac{1}{2\pi}\left(\int_{-\infty}^{\infty} \|H(j\omega)\|^2 d\omega\right)^{1/2}$	$\int_{-\infty}^{\infty} \|Ce^{At}B\| dt$

If we let $h(t) = Ce^{At}B$ be the impulse response for the system and define $\|H\|_\infty = \sup_\omega |H(j\omega|$ and $\|H\|_2 = \frac{1}{2\pi}\left(\int_{-\infty}^{\infty} |H(j\omega)|^2 d\omega\right)^{1/2}$ then

	$\|u\|_2$	$\|u\|_\infty$
$\|y\|_2$	$\|H\|_\infty$	∞
$\|y\|_\infty$	$\|H\|_2$	$\|h\|_1$

These define the three major approaches to linear control:

- $H_\infty = L_2$ to L_2
- $H_2 = L_2$ to L_∞
- $l_1 = L_\infty$ to L_∞

Remarks:

1. These system norms only work for *linear systems* since otherwise looking at $\|u\| \leq 1$ is not sufficient.
2. *Can* extend to nonlinear systems if you constrain the inputs to be $u \in U = \{u : \|u\|_2 \leq 1\}$. Get "nonlinear H_∞" this way.
3. The focus of DFT is H_∞ control $\implies$ $L_2 \to L_2$ induced norm. Think of this as the "energy" of the input (or noise) to the "energy" of the output.
4. Proofs of these norms are given in the book. OK to skip, but here is the idea (for H_∞):
   Let $\hat{y}(s) = \int_{-\infty}^{\infty} y(t)e^{st}dt$ (Laplace transform)
   Can show that $\|y\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{y}(j\omega)|^2 d\omega$ (Parseval's theorem)
   Can also show that $\hat{y}(s) = H(s)u(s)$ (convolution $\to$ multiplication)
   $$\begin{aligned}\|y\|_2^2 = \|\hat{y}\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{H}(j\omega)|^2|\hat{u}(j\omega)|^2 d\omega \\ \leq \|H\|_\infty^2 \cdot \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{u}(j\omega)|^2 d\omega = \|H\|_\infty^2 \|u\|_2^2\end{aligned}$$
   $\implies \|y\|_2 \leq \|H\|_\infty \|u\|_2$.
   To complete the proof, show that there exists $u(t)$ that achieves this bound so that
   $$\max_{\|u\|_2 \leq 1} \|y\|_2 = \max_{\|u\|_2 \leq 1} \|Hu\|_2 = \|H\|_\infty$$

5 Stability and Performance

We now show how to write conditions for stability and performance in terms of the norms of systems. We assume that the process and controller are LIT systems. In addition, we will make the simplifying assumption that the disturbances and noise enter linearly at the same point as the inputs and outputs of the process (as shown in Figure 1 on page 2).

Remarks:

1. $H(s)$ is the Laplace transform of the impulse response $h(t)$

2. $H(s)$ is rational if the system is finite dimensional

$$H(s) = \frac{n(s)}{d(s)} \qquad n(s), d(s) \text{ polynomials}$$

3. Standard terms:

   - $H(s)$ is *stable* if it is analytic in the closed right half plane (i.e., no RHP poles)
   - $H(s)$ is *proper* if $H(j\infty)$ is finite ($\deg d \geq \deg n$)
   - $H(s)$ is *strictly proper* if $H(j\infty) - 0$ ($\deg d > \deg n$)
   - $H(s)$ is *biproper* if $H(s)$ and $H^{-1}(s)$ are proper ($\deg d = \deg n$)

4. Parseval's theorem

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} |H(j\omega)|^2 d\omega = \int_{-\infty}^{\infty} |h(t)|^2 d\omega$$

Stability

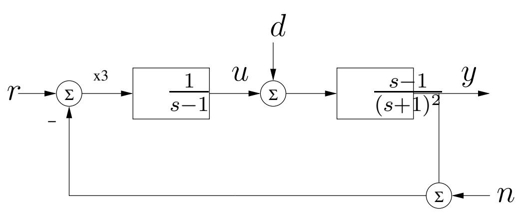
Proposition 2. *If a transfer function is stable then bounded inputs yield bound outputs.*

Proof. Assume that H is stable. Then $\|h\|_1$ is bounded and hence $\|y\|_\infty \leq \|h\|_1 \|u\|_\infty \implies$ bounded inputs yield bounded outputs. The converse is also true. □

Definition 6. A feedback interconnection is *internally stable* if the closed loop transfer function from any input to any other signal is stable.

Remarks:

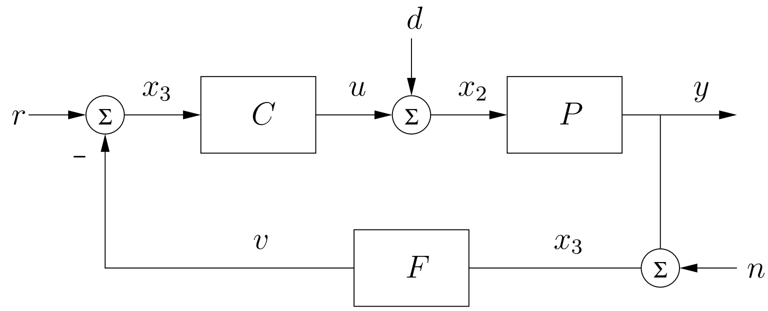
1. Internal stability $\implies$ signals inside the feedback loop remain bounded. Example of something that is *not* internally stable:



$$H_{yr} = \frac{PC}{1+PC} = \frac{1}{(s+1)^2} \quad \text{(work out)}$$

$$H_{ur} = \frac{C}{1+PC} = \frac{(s+1)^2}{(s-1)(s+1)^2} \quad \text{(unstable)}$$

2. Consider the basic feedback loop



$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{1+PCF} \begin{bmatrix} 1 & -PF & -F \\ C & 1 & -CF \\ PC & P & 1 \end{bmatrix} \begin{bmatrix} r \\ d \\ n \end{bmatrix}$$

Internally stable $\iff$ all 9 transfer functions are stable (other signals are simple linear combinations, eg $y = x_3 - n$, etc).

3. If $F = 1$, can reduce the number of transfer functions which need to be checked (homework).

If P, C and F are *rational* transfer functions, we can write them as

$$P = \frac{n_P}{d_P} \qquad C = \frac{n_C}{d_C} \qquad F = \frac{n_F}{d_F}$$

where each n_X and d_X re *coprime* polynomials (no common zeros).

Note that the closed loop poles of the basic feedback loop are given by the zeros of $n_P n_C n_F + d_P d_C d_F = \lambda(s) =$ *characteristic polynomial.*

Theorem 3. *The basic feedback loop is internally stable if and only if there are no zeros of* $\lambda(s) = n_P n_C n_F + d_P d_C d_F$ *in the closed right half plane.*

Proof. Consider the case when $F = 1$ (for simplicity). Can show

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{n_P n_C + d_P d_C} \begin{bmatrix} d_P d_C & -n_P d_C & -d_P d_C \\ d_P n_C & d_P d_C & -d_P n_C \\ n_P n_C & n_P d_C & d_P d_C \end{bmatrix} \begin{bmatrix} r \\ d \\ n \end{bmatrix}$$

and the poles are all contained in the common factor $1/(n_P n_C + d_P d_C)$. □

Theorem 4. *The basic feedback loop is internally stable if and only if the following conditions hold:*

1. *$1 + PCF$ has no zeros in the right half plane ($\mathrm{Re}\, s \geq 0$)*

2. *There are no pole zero cancellations in $\mathrm{Re}\, s \geq 0$ when PCF is formed*

Proof. See DFT □

Performance

We choose our performance goal to try to get good tracking for a *set* of signals.

Sensitivity function: $S = \dfrac{1}{1+PC} = H_{er}$

- We would like to keep this transfer function small ($\implies$ good tracking)
- Assume internal stability $\implies$ S is stable and proper
- Assume P is strickly proper $\implies$ $S(j\infty) = 1$ $\implies$ bad tracking at high frequency.

Complementary sensitivity function: $T == \dfrac{PC}{1+PC} H_{yr}$

- Note that $-T$ is equal to H_{du} $\implies$ gives the transfer function between disturbances and input
- Would like to keep this transfer function small as well
- Can't make *both* S and T small since

$$S + T = \frac{1}{1+PC} + \frac{PC}{1+PC} = 1 \qquad \text{for all } \omega$$

$\implies$ tradeoff between input magnitude and tracking error.

Suppose we want performance to mean small error for unit sized inputs. Try

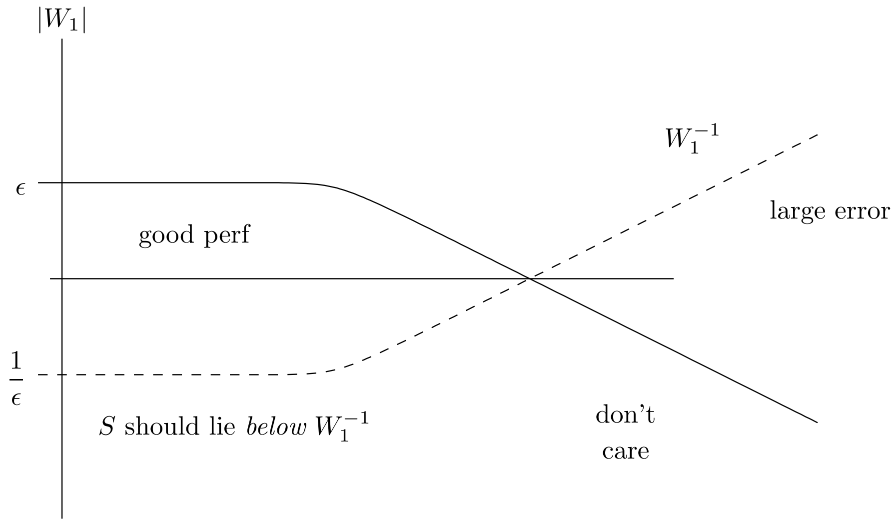
$$\|S\|_\infty \le \epsilon \qquad \epsilon = \text{tolerated error}$$

$\implies$ for $r(t) = \sin(\omega t)$, $|e(t)| \le \epsilon$.

Problem: $S(j\infty) = 1$ $\implies$ $\|S\|_\infty \ge 1$ $\implies$ can't make S arbitarily small.

Fix: Specify performance in terms of *weighted sensitivity function.*

$$\|W_1 S\|_\infty \le 1 \qquad \text{when } W_1 \text{ is small, allow larger errors.}$$



Remarks:

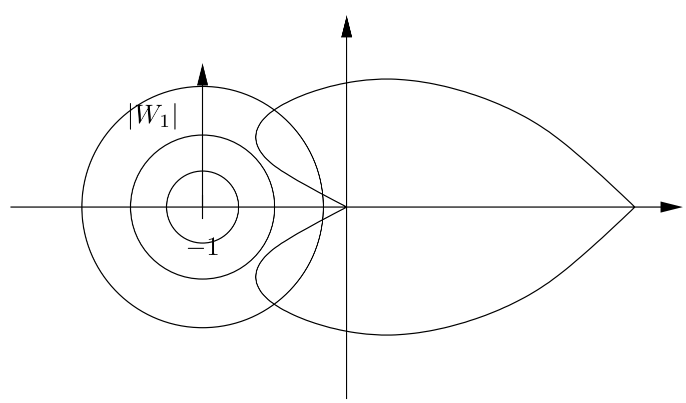
1. Lots of ways of justifying why $\|W_1 S\|_\infty$ makes sense (L_2-induced norm, etc). Big reason: it works well.
2. Other possibilities:

$$\begin{bmatrix} e \\ u \end{bmatrix} = -\begin{bmatrix} PS & S \\ T & CS \end{bmatrix} \begin{bmatrix} d \\ n \end{bmatrix} \qquad \begin{aligned} S &= \text{sensitivity function} \\ T &= \text{complementary sensitivity function} \end{aligned}$$

   Can shoose *any* of these (or combination) for performance spec.

Nyquisti plot interpretation:

$$\begin{aligned} \|W_1 S\|_\infty < 1 &\iff \left| \frac{W_1(j\omega)}{1 + L(j\omega)} \right| < 1 \\ &\iff |W_1(j\omega)| < |1 + L(j\omega)| \leq 1 + |L(j\omega)| \end{aligned}$$



The size of the disk at -1 changes size depending on frequency $\implies$ $|W_1$ gives the distance that we want to stay away from -1 (where we go unstable).