CDS 110b

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Winter 2006   Due: 1 Feb 06

Note: In the upper hand corner of the *second* page of your homework set, please put the number of hours that you spent on this homework set (including reading).

Unless otherwise specified, you may use MATLAB or Mathematica as long as you include a copy of the code used to generate your answer.

1. The linearized equations of motion of a simple pendulum are given by

$$\ddot{\theta} + \omega_0^2 \theta = u$$

   (a) Design a state feedback controller so that the roots of the closed-loop characteristic equation are at $s = -4 \pm 4j$ (assuming you can measure the state directly). Your gains should depend on ω_0.

   (b) Design an estimator (observer) that reconstructs the state of the pendulum given measurements of $\dot{\theta}$. Pick the estimator roots to be at $s = -10 \pm 10j$. Provide an explicit, state space differential equation for your estimator design (again depending on ω_0).

   (c) Setting $\omega_0 = 5$ rad/sec, plot the frequency response for the loop transfer function for the combined estimator, controller and process, and determine the gain and phase margin for the system.

2. (Friedland 2.1, 3.6, 7.2) Consider the inverted pendulum on a cart driven by an electric motor. The linearized equations of motion for the system are given by

$$\ddot{x} + \frac{k^2}{Mr^2R}\dot{x} + \frac{mg}{M}\theta = \frac{k}{MRr}u$$

$$\ddot{\theta} - \left(\frac{M+m}{Ml}\right)g\theta - \frac{k^2}{Mr^2Rl}\dot{x} = -\frac{k}{MRrl}u$$

   where k is the motor torque constant, R is the motor resistance, r is the ratio of the linear forces applied to the cart ($\tau = rf$), and e is the voltage applied to the motor. The following numerical data may be used:

$$m = 0.1 \text{ kg} \qquad M = 1.0 \text{ kg} \qquad l = 1.0 \text{ m} \qquad g = 9.8 \text{ m/s}^2$$
$$k = 1 \text{ V} \cdot \text{s} \qquad R = 100 \text{ } \Omega \qquad r = 0.02 \text{ m}$$

   An observer for the inverted pendulum on a motor-driven cart is to be designed using the measurement of the displacement of the cart ($y = x$).

   (a) Find the matrices A, B, C, and D of the state-space characterization of the system.

   (b) Determine the observer gain for which the observer poles lie in a fourth-order Butterworth pattern of radius 5, i.e., the characterstic equation is to be

$$\left(\frac{s}{5}\right)^4 + 2.613\left(\frac{s}{5}\right)^3 + (2+\sqrt{2})\left(\frac{s}{5}\right)^2 + 2.613\left(\frac{s}{5}\right)^1 + 1 = 0.$$

(c) Plot the response of the observer states to an initial state estimate error of $(1, 1, 1, 1)$.

3. A random variable y is the sum of two independent normally (Gaussian) distributed random variables having means m_1, m_2 and variances σ_1^2, σ_2^2 respectively. Show that the probability density function for y is

$$p(y) = \frac{1}{2\pi\sigma_1\sigma_2} \int_{-\infty}^{\infty} \exp\left\{ -\frac{(y - x - m_1)^2}{2\sigma_1^2} - \frac{(x - m_2)^2}{2\sigma_2^2} \right\} dx$$

and confirm that this is normal (Gaussian) with mean $m_1 + m_2$ and variance $\sigma_1^2 + \sigma_2^2$. (Hint: most of the definitions you need should be in the notes posted on the web.)

4. Find a constant matrix A and vectors F and C such that for

$$\dot{x} = Ax + Fw, \ \ y = Cx$$

the power spectrum of y is given by

$$S(\omega) = \frac{1 + \omega^2}{(1 - 7\omega^2)^2 + 1}$$

Describe the sense in which your answer is unique.