CALIFORNIA INSTITUTE OF TECHNOLOGY
Control and Dynamical Systems

CDS 110b

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Winter 2006 | | Due: 25 Jan 06

Note: In the upper left hand corner of the *second* page of your homework set, please put the number of hours that you spent on this homework set (including reading).

Unless otherwise specified, you may use MATLAB or Mathematica as long as you include a copy of the code used to generate your answer.

1. Consider the optimal control problem for the system

$$\dot{x} = ax + bu \qquad J = \tfrac{1}{2}\int_{t_0}^{t_f} u^2(t)\,dt + \tfrac{1}{2}cx^2(t_f),$$

where $x \in \mathbb{R}$ is a scalar state, $u \in \mathbb{R}$ is the input, the initial state $x(t_0)$ is given, and $a, b \in \mathbb{R}$ are positive constants. We take the terminal time t_f as given and let $c > 0$ be a constant that balances the final value of the state with the input required to get to that position. The optimal is derived in the lecture notes for week 1 and is shown to be

$$\begin{aligned} u^*(t) &= -\frac{2abc\, e^{a(2t_f - t_o - t)}x(t_o)}{2a - b^2c\left(1 - e^{2a(t_f - t_o)}\right)} \\ x^*(t) &= x(t_o)e^{a(t-t_o)} + \frac{b^2c\, e^{a(t_f - t_o)}x(t_o)}{2a - b^2c\left(1 - e^{2a(t_f - t_o)}\right)}\left[e^{a(t_f - t)} - e^{a(t + t_f - 2t_o)}\right]. \end{aligned} \tag{1}$$

Now consider the infinite horizon cost

$$J = \tfrac{1}{2}\int_{t_0}^{\infty} u^2(t)\,dt$$

with $x(t)$ at $t = \infty$ constrained to be zero.

   (a) Solve for $u^*(t) = -bPx^*(t)$ where P is the positive solution corresponding to the algebraic Riccati equation. Note that this gives an explicit feedback law ($u = -bPx$).

   (b) Plot the state solution of the finite time optimal controller for the following parameter values

$$a = 2 \qquad b = 0.5 \qquad x(t_0) = 4$$
$$c = 0.1,\ 10 \qquad t_f = 0.5,\ 1,\ 10$$

   (This should give you a total of 6 curves.) Compare these to the infinite time optimal control solution. Which finite time solution is closest to the infinite time solution? Why?

2. Using the solution given in equation (1), implement the finite-time optimal controller in a receding horizon fashion with an update time of $\delta = 0.5$. Using the parameter values in problem 1(b), Compare the responses of the receding horizon controllers to the LQR controller

you designed for problem 1, from the same initial condition. What do you observe as c and t_f increase?

(Hint: you can write a MATLAB script to do this by performing the following steps:

(i) set $t_0 = 0$

(ii) using the closed form solution for x^* from problem 1, plot $x(t)$, $t \in [t_0, t_f]$ and save $x_\delta = x(t_0 + \delta)$

(iii) set $x(t_0) = x_\delta$ and repeat step (ii) until x is small.)

3. In this problem we will explore the effect of constraints on control of the linear unstable system given by

$$\dot{x}_1 = 0.8x_1 - 0.5x_2 + 0.5u$$
$$\dot{x}_2 = x_1 + 0.5u$$

subject to the constraint that $|u| \leq a$ where a is a postive constant.

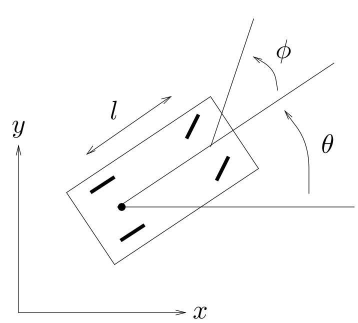
(a) Ignoring the constraint ($a = \infty$) and design an LQR controller to stabilize the system. Plot the response of the closed system from the initial condition given by $x = (1, 0)$.

(b) Use SIMULINK or `ode45` to simulate the the system for some finite value of a with an initial condition $x(0) = (1, 0)$. Numerically (trial and error) determine the smallest value of a for which the system goes unstable.

(c) Let $a_{\min}(\rho)$ be the smallest value of a for which the system is unstable from $x(0) = (\rho, 0)$. Plot $a_{\min}(\rho)$ for $\rho = 1,\ 4,\ 16,\ 64,\ 256$.

(d) *Optional*: Given $a > 0$, design and implement a receding horizon control law for this system. Show that this controller has larger region of attraction than the controller designed in part (b). (Hint: solve the finite horizon LQ problem analytically, using the bang-bang example as a guide to handle the input constraint.)

4. Consider the lateral control problem for an autonomous ground vehicle, as shown below:

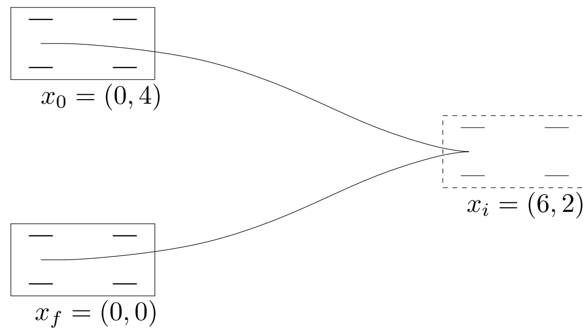


$$\dot{x} = \cos\theta\, v$$
$$\dot{y} = \sin\theta\, v$$
$$\dot{\theta} = \frac{1}{\ell}\tan\phi\, v,$$

Let (x, y, θ) represent the state and (v, ϕ) the inputs. For simplicity we will assume $\ell = 1$ meter.

(a) Show that this system is differentially flat using $z = (x, y)$ and solve explicitly for the state and input in terms of the flat output and its derivatives. Note any special situations in which additional assumptions are required (eg, if you equations go singular at some point, you may need to stay away from these points).

(b) Using the fact that the system is differentially flat, find an explicit trajectory that solves the following parallel parking manuever:



Your solution should consist of two segments: a curve from x_0 to x_i with $v > 0$ and a curve from x_i to x_f with $v < 0$. For the trajectory that you determine, plot the trajectory in the plane (x versus y) and also the inputs v and ϕ as a function of time.