CALIFORNIA INSTITUTE OF TECHNOLOGY
Control and Dynamical Systems

CDS 110b

R. M. Murray — **Problem Set #2** — Issued: 11 Jan 06
Winter 2006 — Due: 18 Jan 06

Note: In the upper left hand corner of the *second* page of your homework set, please put the number of hours that you spent on this homework set (including reading).

Unless otherwise specified, you may use MATLAB or Mathematica as long as you include a copy of the code used to generate your answer.

1. The output $c(t)$ in a position-control system is governed by

$$J\ddot{c} = u,$$

where $u(t)$ is applied force.

(a) Write down a state space realization (find A and B).

(b) Use the matrix Riccati equation to find the feedback control law minimizing

$$\int_0^\infty (c^2 + q^2 u^2) dt.$$

(c) Show that the optimal control system has damping ratio $\frac{1}{\sqrt{2}}$.

(d) What is the corresponding optimal value of natural frequency?

(See AM05, Sec 4.4 if you don't remember how damping ratio (or factor) and natural frequency are defined.)

2. (Friedland 9.6) Consider the dynamics of a DC motor driving an inertial load (see Friedland, page 231 for a picture):

$$\begin{aligned}\dot{\theta} &= \omega \\ \dot{\omega} &= -\alpha\omega + \beta u\end{aligned}$$

where θ is the angular position of the load, ω is the angular velocity, u is the applied voltage, and α and β are constants that depend on the physical parameters of the motor and load. For this problem, let $\alpha = 1$ and $\beta = 3$.

(a) Let $e = \theta - \theta_d$. For the performance criterion

$$V = \int_0^\infty (q_1^2 e^2 + u^2)\, d\tau$$

find and tabulate the control gains and corresponding closed-loop poles for $q_1 = 0.1, 1, 10$.

(b) Plot the transient response (e as a function of t) for the inital error of unity for the values of q_1 in part (a). (Note: you should use the MATLAB `initial` function to get the transient response to an initial error. Set the initial condition for θ appropriately.)

(c) In addition to weighting the position error it is also desired to limit the velocity by using a performance criterion

$$V = \int_0^\infty (q_1^2 e^2 + q_2^2 \dot{e}^2 + u^2)\, d\tau.$$

For the values of q_1^2 used in part (a) and $q_2^2 = 0.1q_1^2, q_1^2, 10q_1^2$ find the control gains and corresponding closed loop poles.

(d) Plot the transient response as in part (b) for a range of q_1^2 and q_2^2 (you need not include all 9 plots; just the "interesting" ones). Compare the results with those of part (b). Are the results as expected?

3. (Friedland 2.1, 3.6, 7.2, 9.10) Consider the motor-driven inverted pedulum on a cart, whose linearized dynamics are given by

$$\ddot{x} + \frac{k^2}{Mr^2R}\dot{x} + \frac{mg}{M}\theta = \frac{k}{MRr}u$$
$$\ddot{\theta} - \left(\frac{M+m}{Ml}\right) g\theta - \frac{k^2}{Mr^2Rl}\dot{x} = -\frac{k}{MRrl}u$$

where k is the motor torque constant, R is the motor resistance, r is the ratio of the linear forces applied to the cart ($\tau = rf$), and u is the voltage applied to the motor. The following numerical data may be used:

$$m = 0.1 \text{ kg} \qquad M = 1.0 \text{ kg} \qquad l = 1.0 \text{ m} \qquad g = 9.8 \text{ m/s}^2$$
$$k = 1 \text{ V}\cdot\text{s} \qquad R = 100\ \Omega \qquad r = 0.02 \text{ m}$$

We wish to optimize the gains using a performance criterion of the form

$$V = \int_0^\infty \left(q_1^2 x^2 + q_3^2 \theta^2 + \rho^2 u^2\right)\, dt$$

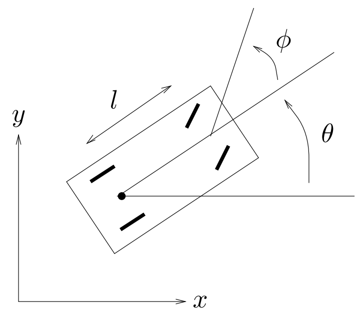
A pendulum angle much greater than 1 degree = 0.017 rad would be precarious. Thus a heavy weighting error on θ is indicated: $q_3^2 = 1/(0.017)^2 \approx 3000$. For the physical dimensions of the system, a position error of the order of 10 cm = 0.1 m is not unreasonable. Hence $q_1^2 = 1/(0.1)^2 = 100$.

(a) Using these values of q_1^2 and q_3^2, determine and plot the gain matrices and corresponding closed loop poles as a function of the control weighting parameter ρ^2 for $0.001 < \rho^2 < 50$.

(b) Repeat part (a) for a heavier weighting: $q_1^2 = 10^4$ on the cart displacement.

(c) Plot the step responses for the controllers you defined in parts (a) and (b) and explain their behavior in terms of the cost functions you used.

4. Consider the lateral control problem for an autonomous ground vehicle, as shown below.



$$\dot{x} = \cos\theta\, v$$
$$\dot{y} = \sin\theta\, v$$
$$\dot{\theta} = \frac{1}{\ell}\tan\phi\, v,$$

The dynamics for the system are given in the equation above, where (x, y) is the location of the center of the rear wheels of the vehicle, θ is the angle of the vehicle with respect to the x axis, v is the forward velocity of the vehicle, ϕ is the angle of the steering wheel (an input) and ℓ is the wheelbase of the vehicle.

We assume that we are given a reference trajectory $r = (x_d, y_d)$ corresponding to the desired trajectory of the vehicle. For simplicity, we will assume that we wish to follow a straight line in the x direction at a constant velocity $v_d > 0$ and hence we focus on the y and θ dynamics:

$$\dot{y} = \sin\theta\, v_d$$
$$\dot{\theta} = \frac{1}{\ell}\tan\phi\, v_d.$$

We let $v_d = 10$ m/s and $\ell = 2$ m.

(a) Design an LQR controller that stabilizes the position y to the origin. Plot the step and frequency response for your controller and determine the overshoot, rise time, bandwidth and phase margin for your design. (Hint: for the frequency domain specifications, break the loop just before the process dynamics and use the resulting SISO loop transfer function.)

(b) Suppose now that $y_d(t)$ is not identically zero, but is instead given by $y_d(t) = r(t)$. Modify your control law so that you track $r(t)$ and demonstrate the performance of your controller on a "slalom course" given by a sinusoidal trajectory with magnitude 1 meter and frequency 1 Hz.