CALIFORNIA INSTITUTE OF TECHNOLOGY
Control and Dynamical Systems

CDS 110b

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Note: In the upper left hand corner of the *second* page of your homework set, please put the number of hours that you spent on this homework set (including reading).

1. An important use of optimal control is to compute a trajectory between two points. In this problem, we consider a different way of solving this problem using the concept of "differential flatness".

   Consider a single input system in reachable canonical form

$$\frac{dx}{dt} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & \ddots & \\ 0 & & & & 1 \\ -a_1 & -a_2 & \dots & & -a_n \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u. \qquad (1)$$

   Suppose that we wish to find an input u that moves the system from x_0 to x_f; this is called the *trajectory generation* problem.

   Notice that if we specify $x_1(t)$ then we can solve equation (1) to obtain the remaining states and inputs that satisfy the dynamics of the system. This allows us to parameterize the solution by a curve of the form

$$x_1(t) = \sum_{k=0}^{N} \alpha_k t^k, \qquad (2)$$

   where N is a sufficiently large integer. Systems in which feasible trajectories for the system can be computed algebraically in this fashion are called *differentially flat.*

   Use these concepts to solve the following:

   (a) Compute the state space trajectory $x(t)$ and input $u(t)$ corresponding to equation (2) and satisfying the differential equation (1). Your answer should be an equation similar to equation (2) for each state x_i and the input u.

   (b) Show that if N is chosen appropriately, the trajectory generation problem can be converted to solving a matrix linear equation for the coefficients $\alpha \in \mathbb{R}^{N+1}$. (Hint: start with $n = 2$ to see how this goes and then solve the general case.)

2. In this problem, you will use the maximum principle to show that the shortest path between two points is a straight line.

   We model the problem by constructing a control system

$$\dot{x} = u$$

where $x \in \mathbb{R}^2$ is the position in the plane and $u \in \mathbb{R}^2$ is the velocity vector along the curve. Suppose we wish to find a curve of minimal length connecting $x(0) = x_0$ and $x(1) = x_f$. To minimize the length, we minimize the integral of the velocity along the curve,

$$J = \int_0^1 \sqrt{\|\dot{x}\|}\, dt,$$

subject to to the initial and final state constraints. Use the maximum principle to show that the minimal length path is indeed a straight line at maximum velocity. (Hint: minimizing $\sqrt{\|\dot{x}\|}$ is the same as minimizing $\dot{x}^T\dot{x}$; this will simplify the algebra a bit.)

3. Consider the problem of moving a two-wheeled mobile robot (eg, a Segway) from one position and orientation to another. The dynamics for the system is given by the nonlinear differential equation

$$\begin{aligned}\dot{x} &= \cos\theta\, v\\ \dot{y} &= \sin\theta\, v\\ \dot{\theta} &= \omega\end{aligned}$$

where (x, y) is the position of the rear wheels, θ is the angle of the robot with respect to the x axis, v is the forward velocity of the robot and ω is spinning rate. We wish to choose an input (v, ω) that minimizes the time that it takes to move between two configurations (x_0, y_0, θ_0) and (x_f, y_f, θ_f), subject to input constraints $|v| \leq L$ and $|\omega| \leq M$.

Use the maximum principle to show that any optimal trajectory consists of segments in which the robot is traveling at maximum velocity in either the forward or reverse direction, and going either straight, hard left ($\omega = -M$) or hard right ($\omega = +M$).

Note: one of the cases is a bit tricky and can't be completely proven with the tools we have learned so far. However, you should be able to show the other cases and verify that the tricky case is possible.