CDS 110b: Lecture 8-1
Robust Stability

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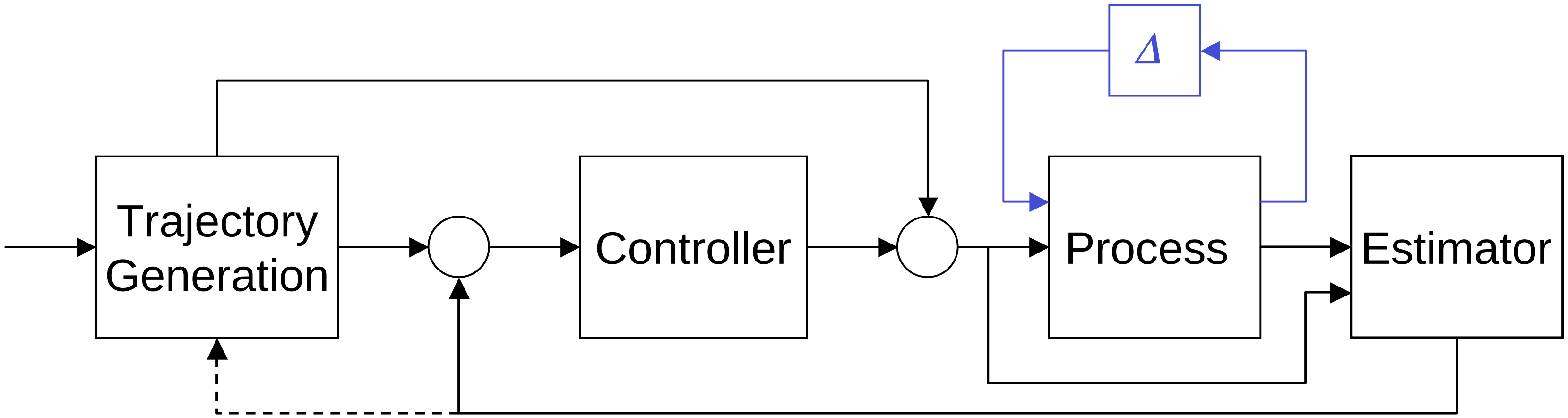
Goals:

- Describe methods for representing unmodeled dynamics
- Derive conditions for robust stability

Reading:

- DFT, Sections 4.1-4.3

Game Plan: Robust Performance



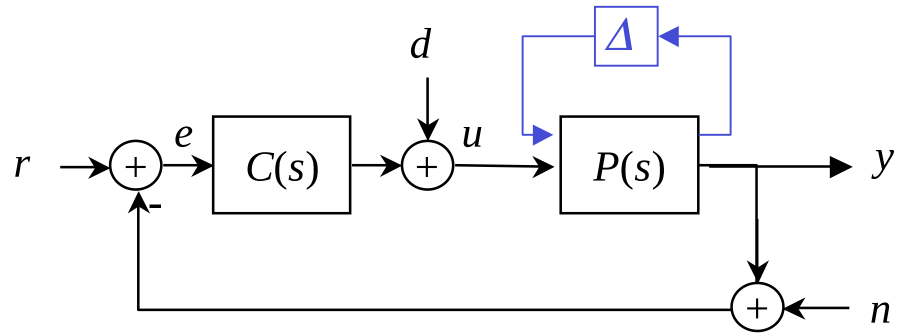
Robust Control

- Stability: bounded inputs $\rightarrow$ bounded outputs
- Performance: keep errors small

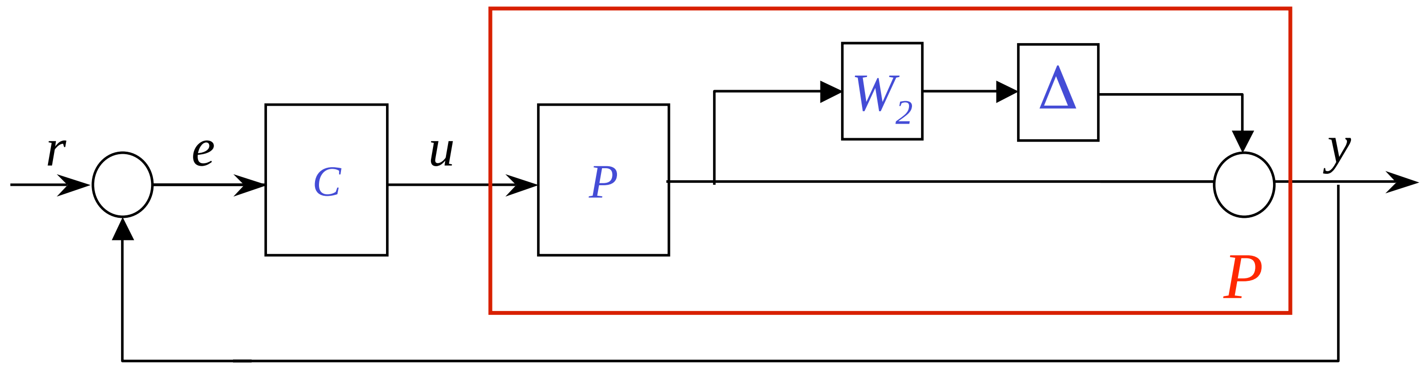
Robustness: do these things for all $\|\Delta\|_\infty < 1$

Simplifying case: focus on basic control loop

- Stability = internal stability
- Performance = weighted sensitivity
- Robustness = ??



Robust Stability



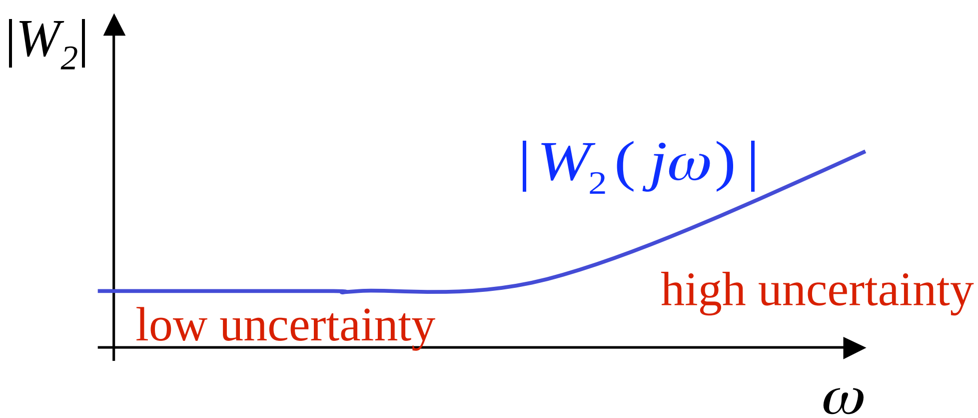
Multiplicative Uncertainty

- Model plant as nominal with additional dynamics given by $W_2\,\Delta$

$$\tilde{P} = P(1 + W_2\Delta)$$

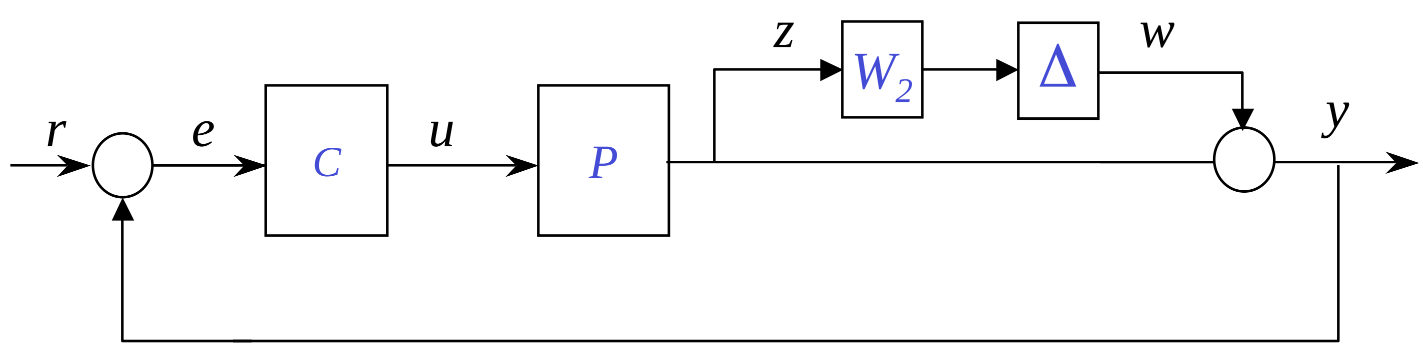
W_2 = frequency weight

Δ = uncertainty; require $|\Delta(jw)| \le 1$



- Δ allows *any* dynamics to be inserted into the plant
- Can be used to model parameter uncertainty or unmodeled dynamics

Complementary Sensitivity and Robustness



Thm *A controller C provides robust stability to multiplicative perturbations if and only if*

$$\boxed{|W_2(j\omega)\, T(j\omega)| \leq 1} \quad \textit{for all } \omega.$$

where

Complementary sensitivity function

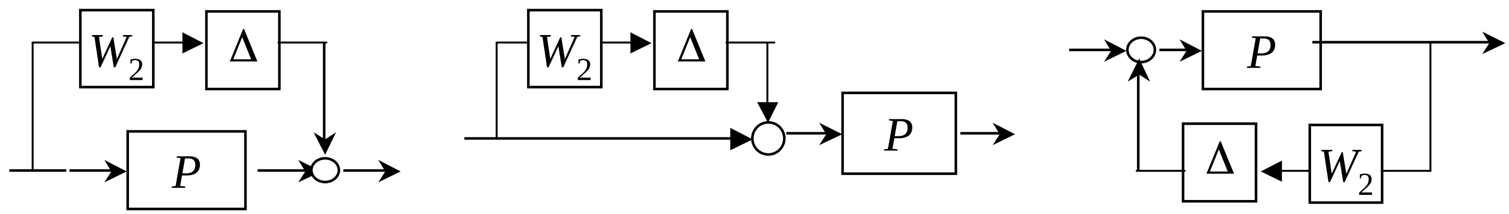
$$\left. T := \frac{PC}{1+PC} = H_{zw} \right\}$$

Intuition: H_{zw} represents the transfer function seen by the weighted uncertainty $W_2\,\Delta$

Note: this theorem guarantees stability for any *transfer function* $\Delta(s)$ satisfying $|\Delta(j\omega)| < 1 \Rightarrow$ allows unmodeled *dynamics* (as well as parametric uncertainty)

Models for Uncertainty

Additive uncertainty | Multiplicative uncertainty | Feedback uncertainty



Each model describes a class of process dynamics:

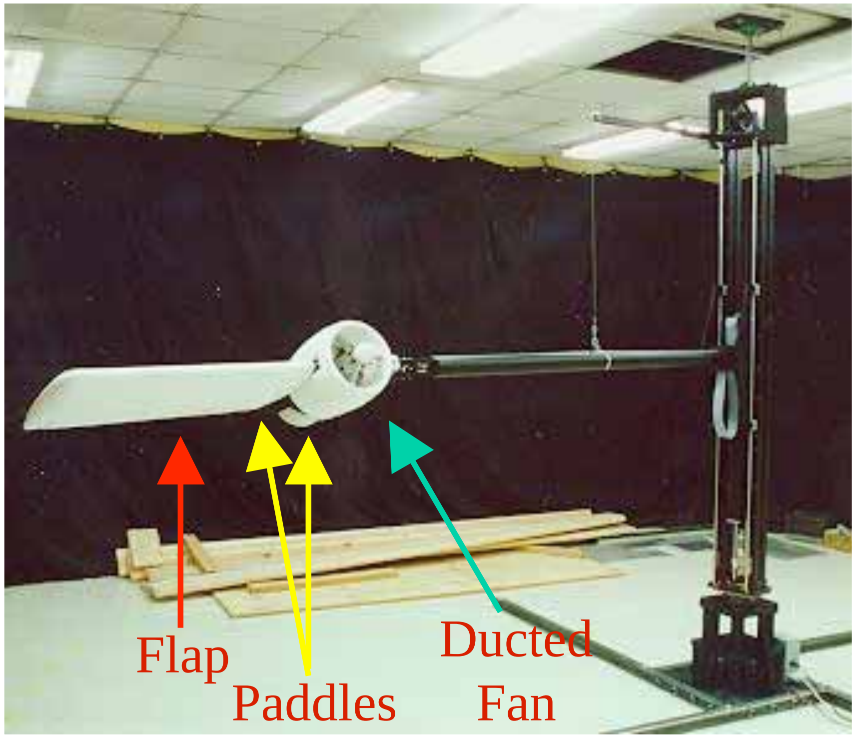
- Additive: $\tilde{P} = P + W_2\Delta$
- Multiplicative: $\tilde{P} = P(1 + W_2\Delta)$
- Feedback: $\tilde{P} = P/(1 + PW_2\Delta)$

Use $||W_2||$ to shape the unmodeled dynamics; $||\Delta||_\infty < 1$ in all cases

Robust stability conditions given by small gain theorem

- Compute transfer function around Δ block and require that this be < 1
- (If not, can choose Δ with $||\Delta||_\infty \leq 1$ to destabilize)

Example: Caltech Ducted Fan



Caltech ducted fan:

- Replicates longitudinal dynamics of an aircraft
- Controlled by dSPACE real-time control system
  - Two TI C40 DSPs + two DEC Alpha processors
  - Actuation: fan, wing flap, thrust vectoring
  - Sensing: optical encoders on stand

Goal: track *aggressive* trajectories with high accuracy in presence of *uncertainties*

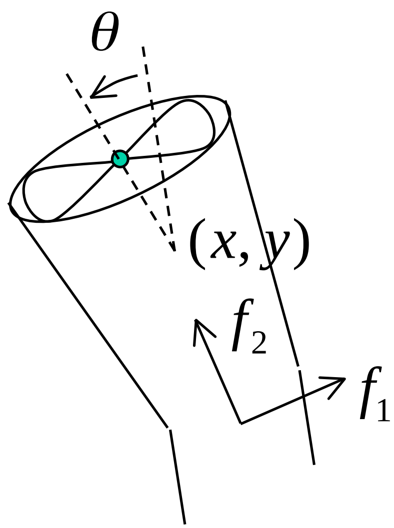
Performance specs

- Stable operation
- Speed and accuracy

Uncertainty specs

- Noise: wind (wall effects), electrical noise
- Model uncertainty: variable motor dynamics, inertial parameters, aerodynamic variations, etc

Caltech Ducted Fan Model



$$m\ddot{x} = f_1 \cos\theta - f_2 \sin\theta$$
$$m\ddot{y} = f_1 \sin\theta + f_2 \cos\theta - mg$$
$$J\ddot{\theta} = rf_1 - d\dot{\theta} - mgl\sin\theta$$

Nonlinear Differential Equations

$$\boxed{\frac{\theta(s)}{u(s)} = \frac{r}{Js^2 + ds + mgl}}$$

Transfer Function (side force to pitch)

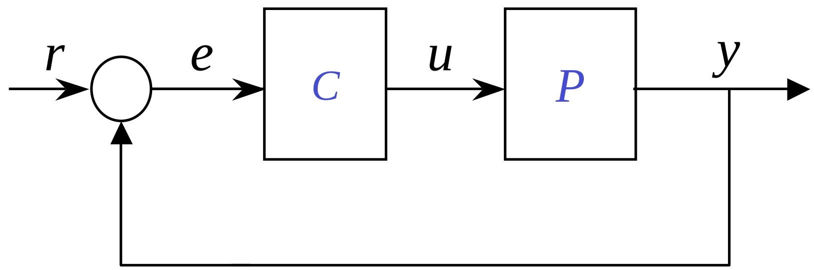
Parameter values

J	0.0475	inertia around pitch axis, kg m^2
m	1.5	mass of fan, kg
r	0.25	distance to flaps, m
g	10	gravitational constant, m/sec^2
d	0.1	damping factor (estimated)
l	0.05	offset of center of mass, m

Sources of uncertainty

- Unknown force location, r
- Unknown damping coefficient, d
- Unmodelled aerodynamics
- Unmodelled stand deflections
- Sensor noise (x, y, θ)

Step 1: Nominal Performance



Reasonable performance spec – keep H_{er} small

- Motivation: small changes in reference generate small errors
- Can also show that this minimizes the effect of external disturbances

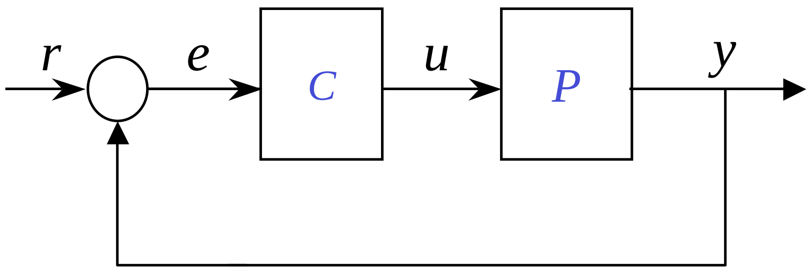
Q: How do we specify the "size" of H_{er}

A: Use magnitude of Bode plot = sinusoidal response

$$H_{er} = \frac{1}{1+PC} =: S$$

Sensitivity function

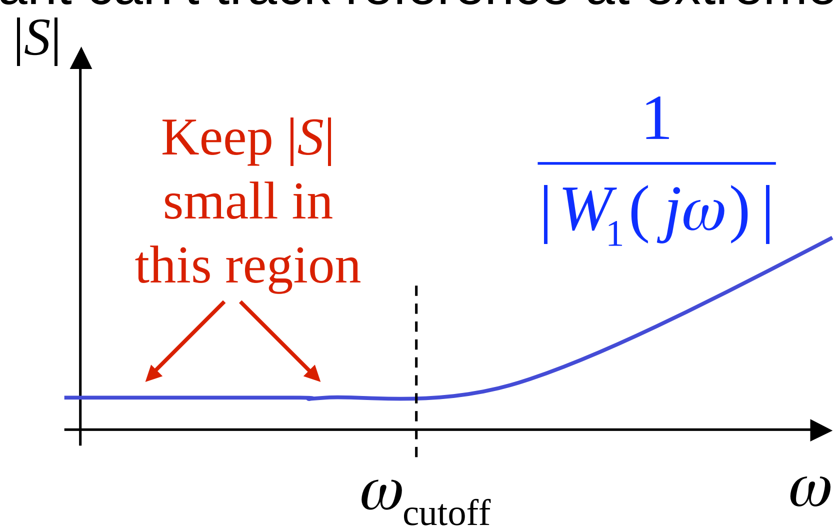
Performance Specification via Weighted Sensitivity



$$H_{er} = \frac{1}{1+PC} =: S$$

Use frequency weighting to specify performance over given range

- Doesn't make sense to ask for small error over *all* frequencies
- Plant can't track reference at extremely high frequencies

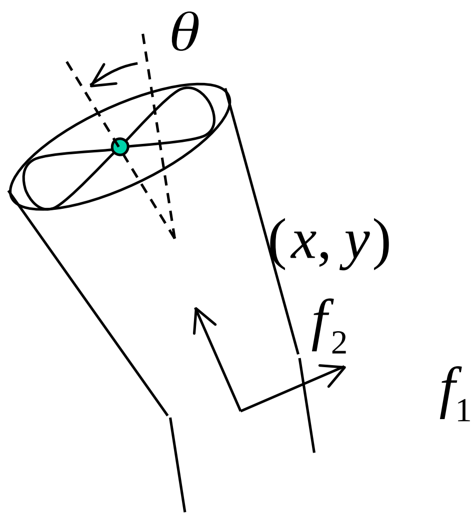


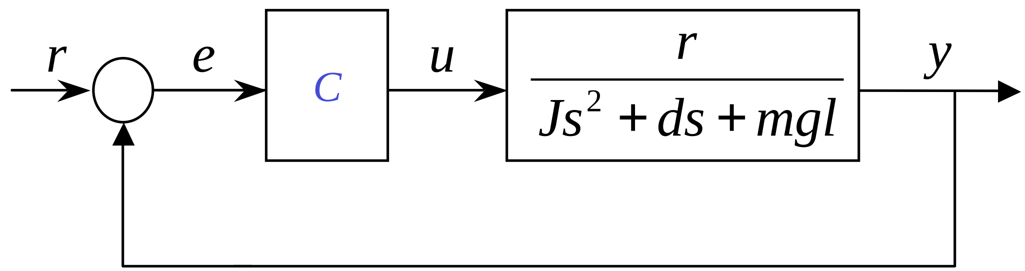
$$|S(j\omega)| \le \frac{1}{|W_1(j\omega)|}$$

$$\Updownarrow$$

$$\boxed{|W_1(j\omega)S(j\omega)| \le 1}$$

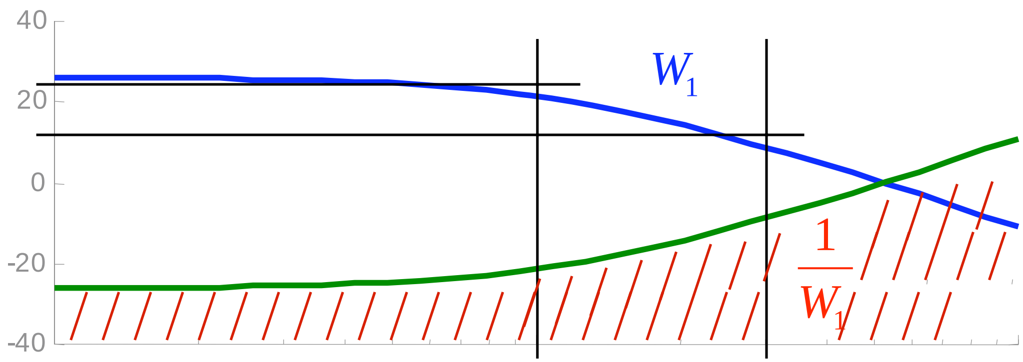
Example: Caltech Ducted Fan Performance (1/2)





Performance specification on H_{er}

- Less than 5% tracking error up to 2 Hz
- Less than 10% tracking error up to 5 Hz



20 log(0.05)

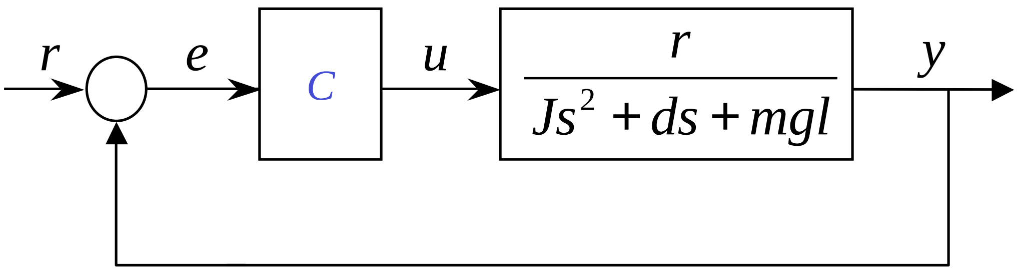
5% tracking error

$$|S(j\omega)| \le \frac{1}{|W_1(j\omega)|}$$

$$\Updownarrow$$

$$|W_1(j\omega)S(j\omega)| \le 1$$

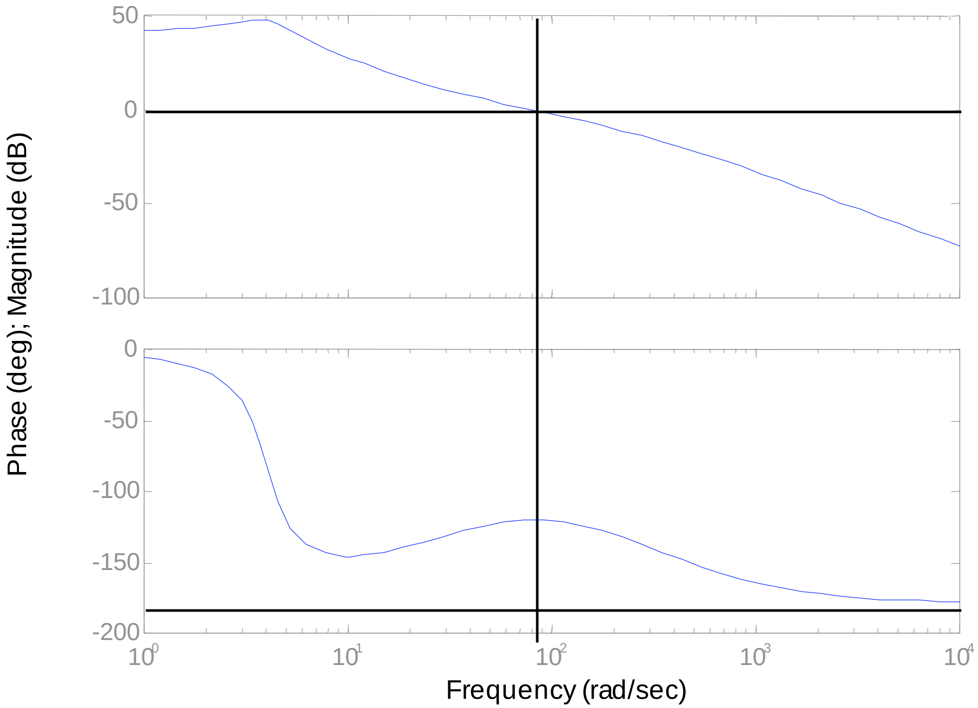
Example: Caltech Ducted Fan Performance (2/2)



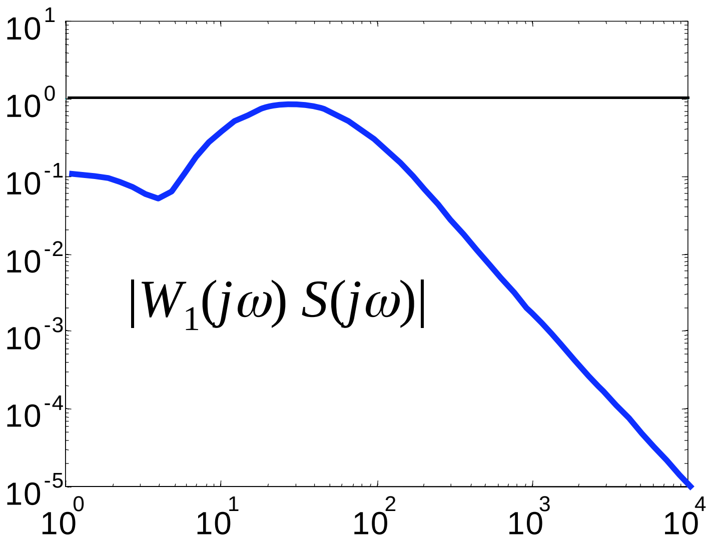
$$C(s) = 20\frac{(s+25)}{(s+300)}$$

$$W_1 = \frac{20}{(s/12+1)^2}$$

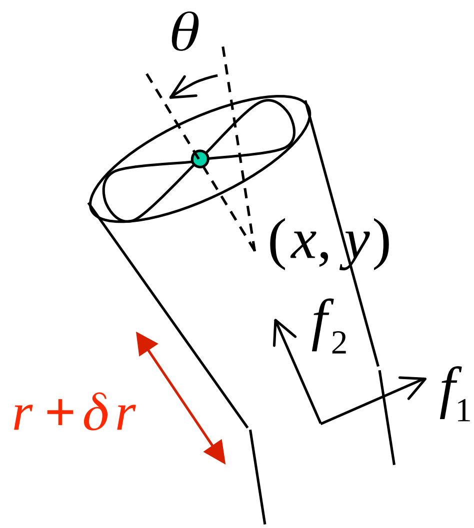
Bode Plot



Q: Does system satisfy performance specification?



Example: Caltech Ducted Fan Robustness (1/2)

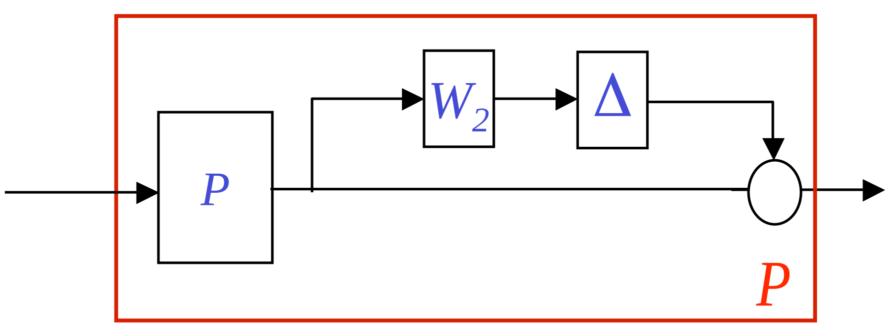


$$P = \frac{\theta(s)}{u(s)} = \frac{r + \delta r}{Js^2 + ds + mgl}$$

Robustness specification on *P*

- Allow up to 20% variation in location of force vector, *r*
- Force location can change *dynamically*

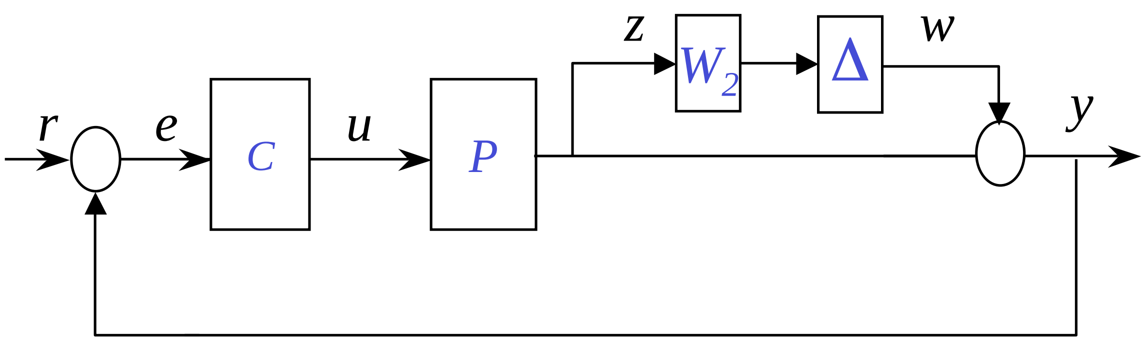
Rewrite uncertainty specification in terms of Δ block:



$$\begin{aligned} P &= \frac{r + \delta r}{Js^2 + ds + mgl} \\ &= \frac{r}{Js^2 + ds + mgl} + \frac{\delta r}{Js^2 + ds + mgl} \\ &= P + \frac{\delta r}{r} P = P(1 + \underbrace{\frac{\delta r}{r}}_{W_2\,\Delta}) \end{aligned}$$

Choose $W_2 = 0.2$, with $|\Delta(s)| < 1$ $\longrightarrow$ $W_2\,\Delta$

Example: Caltech Ducted Fan Robustness (2/2)



$$P(s) = \frac{r}{Js^2 + ds + mgl}$$

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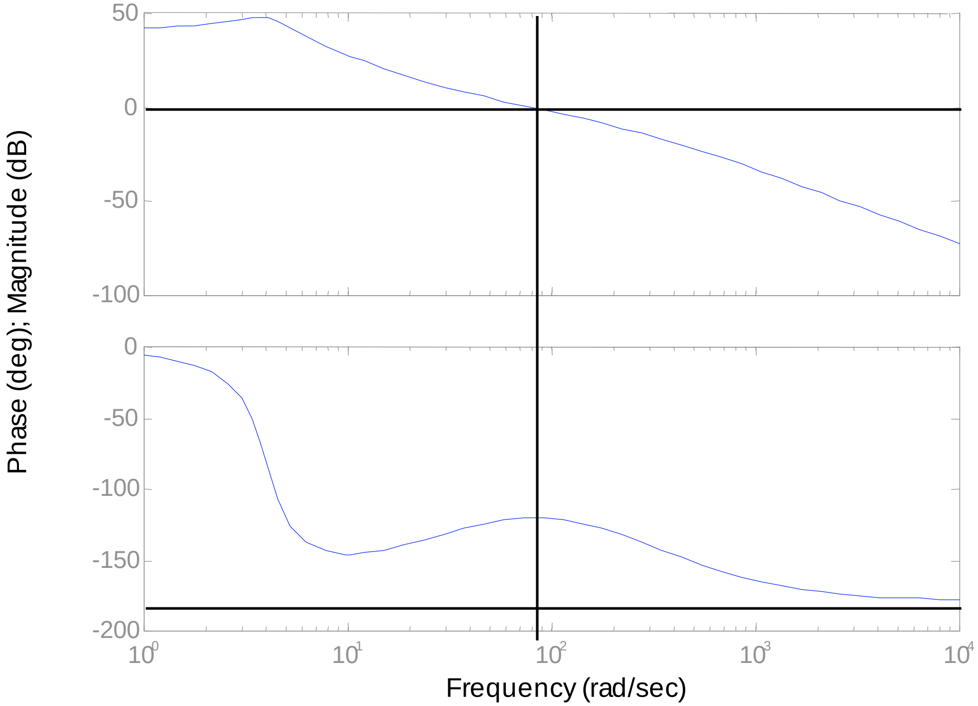
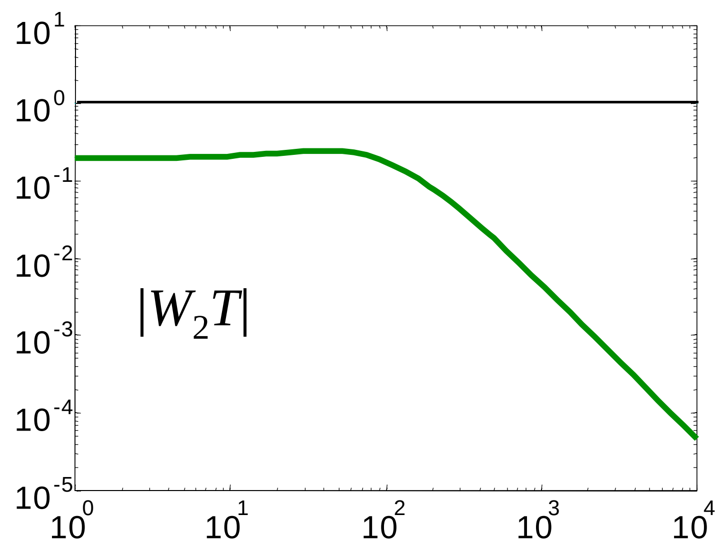
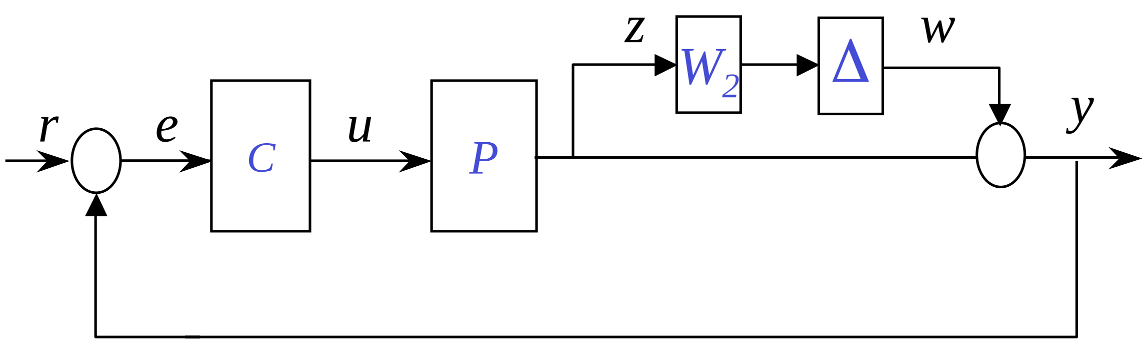
Bode Plot



Q: System is *nominally* stable, but is it *robustly* stable?



Preview: Robust *Performance*



$$|S(j\omega)| \le \frac{1}{|W_1(j\omega)|}$$

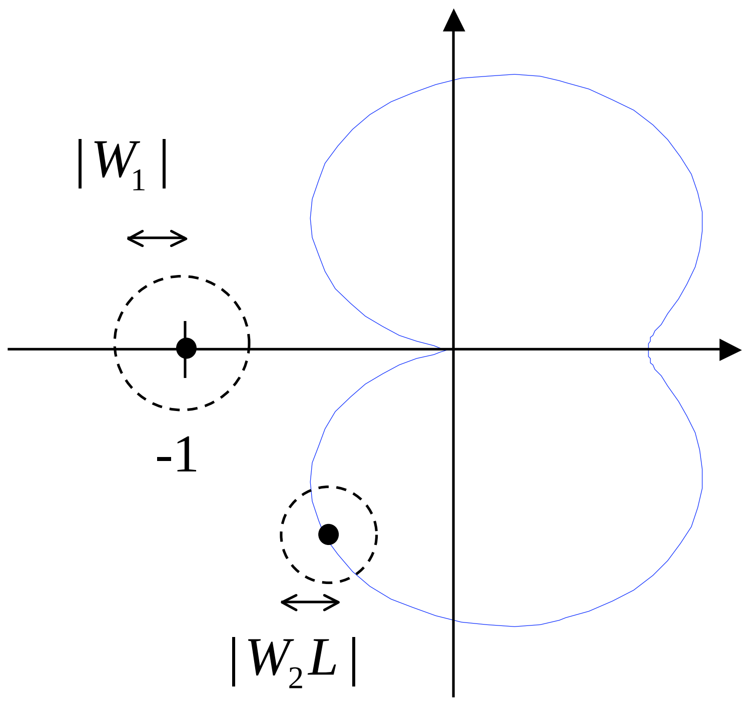
Thm *C* provides *robust performance* to multiplicative uncertainty if and only if

$$\max_{\omega}(|W_1 S| + |W_2 T|) < 1$$

Remarks

- Gives conditions for *guaranteed* robust performance
- Given W_1 and W_2, still need to find C that works

Nyquist Interpretation of Robust Performance



Stability and performance conditions:

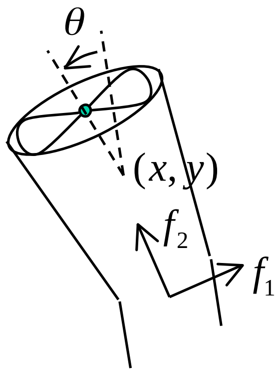
1. Nyquist plot outside –1
   $\Rightarrow$ nominal stability
2. Nyquist plot outside W_1 ball
   $\Rightarrow$ nominal performance
   (generalizes gain/phase margin)
3. Nyquist plot outside W_2 ball
   $\Rightarrow$ robust stability
4. Combined non-intersection
   $\Rightarrow$ robust performance

Notes

- Size of balls varies as frequency changes
- Condition is equivalent to

$$\max_{\omega}(|W_1S|+|W_2T|)<1$$

Example: Caltech Ducted Fan Design Problem (1/2)



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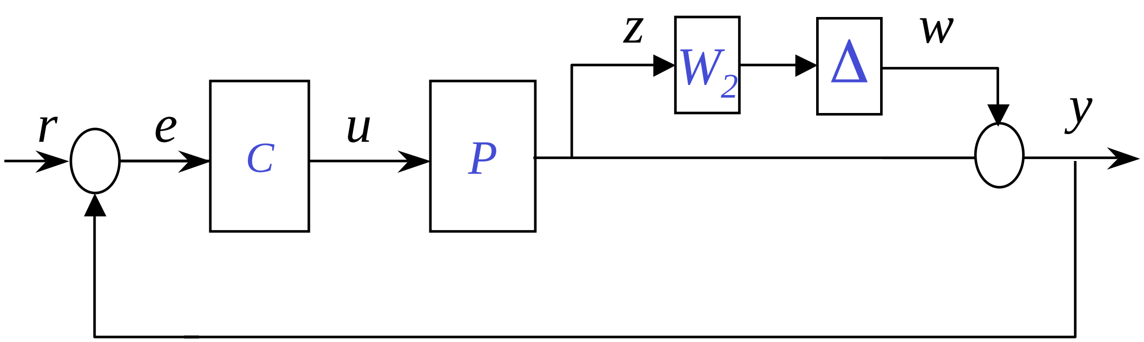
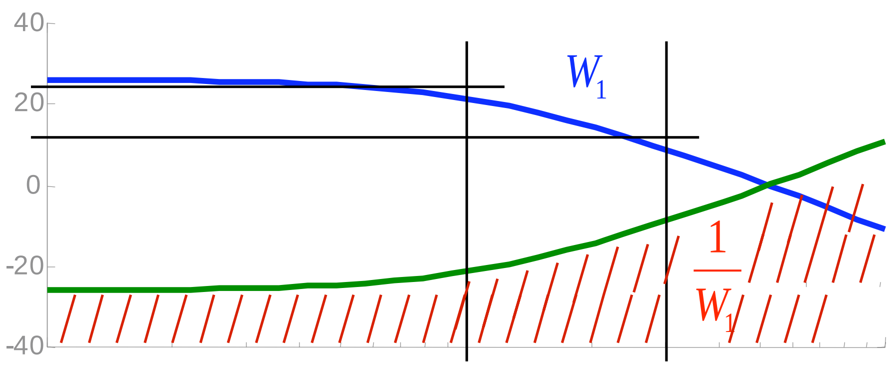
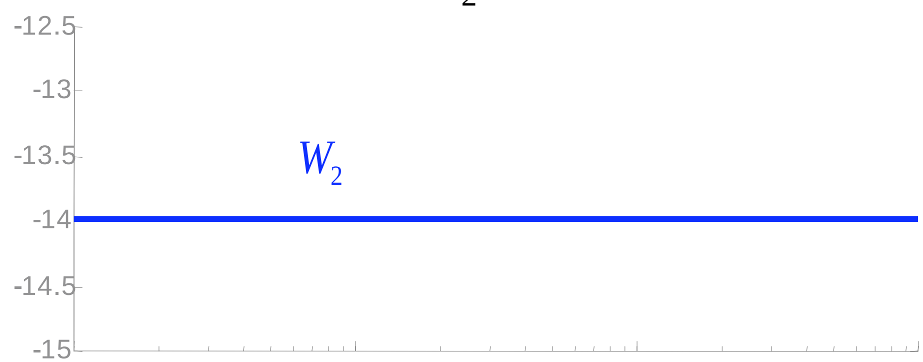


$$|S(j\omega)| \le \frac{1}{|W_1(j\omega)|}$$

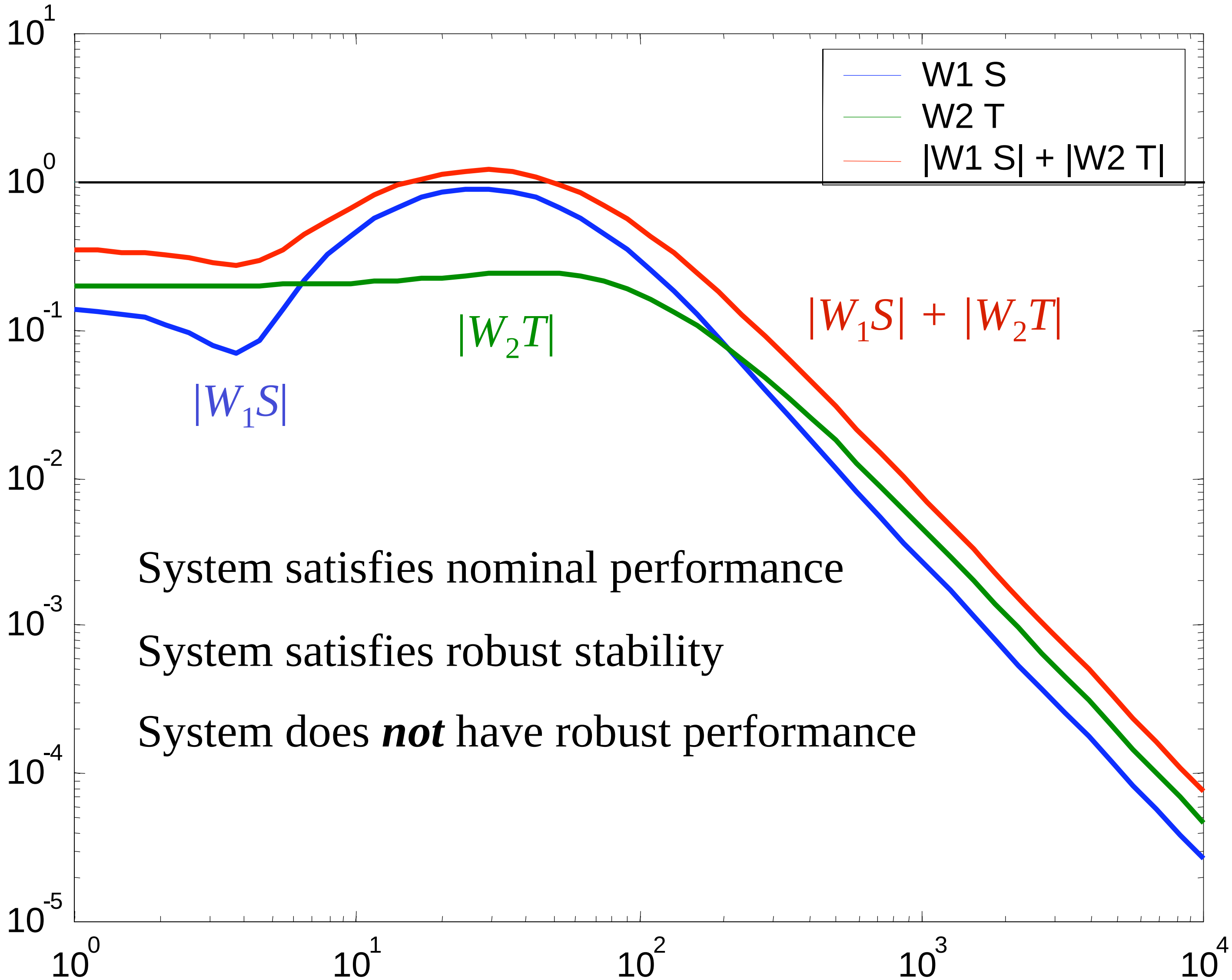
$$W_1 = \frac{20}{(s/12+1)^2}$$

$$W_2 = 0.2$$





Example: Caltech Ducted Fan Design Problem (2/2)

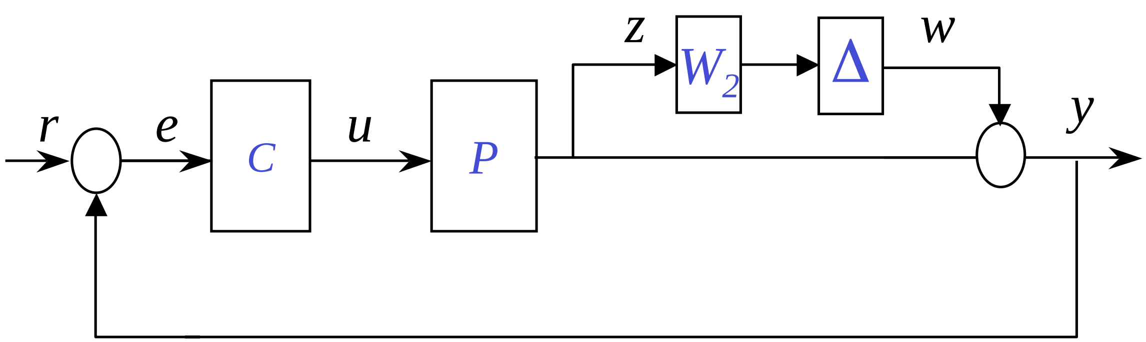


System satisfies nominal performance

System satisfies robust stability

System does ***not*** have robust performance

Summary



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