CDS 110b: Lecture 6-1
Sensor Fusion

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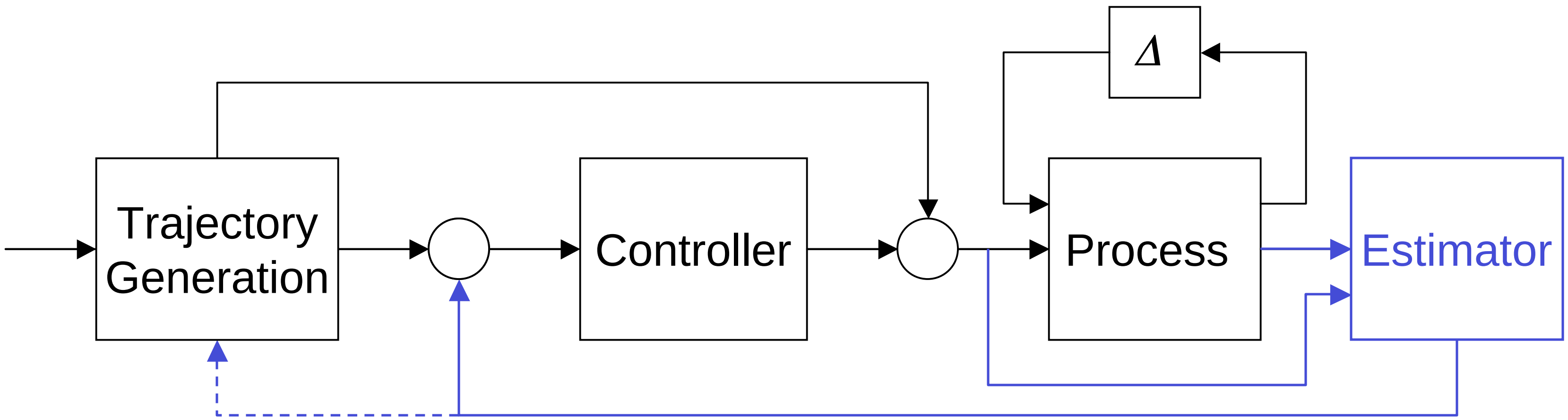
Goals:

- Describe the use of Kalman filtering for sensor fusion
- Give extensions of Kalman filters to nonlinear systems, discrete time systems

Reading:

- Friedland, Chapter 11

Kalman Filter



System dynamics: linear process + Gaussian white noise

$$\dot{x} = Ax + Bu + Fv \qquad E\{v(s)v^T(t)\} = Q(t)\delta(t-s)$$
$$y = Cx + w \qquad E\{w(s)w^T(t)\} = R(t)\delta(t-s)$$

Time-varying statistics OK

Estimator: prediction + correction

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$$

$$L(t) = P(t)C^T R^{-1}$$
$$P(t) = E\{(x(t) - \hat{x}(t))(x(t) - \hat{x}(t))^T\}$$

Covariance update

$$\dot{P} = AP + PA^T - PC^T R^{-1}(t)CP + FQ(t)F^T$$
$$P(0) = E\{x(0)x^T(0)\}$$

Variation #1: Sensor Fusion

What happens if we have redundant sensors?

- Kalman filter "fuses" data measurements according to covariance

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x}) \qquad \begin{aligned} L(t) &= P(t)C^T R^{-1} \\ P(t) &= E\{(x(t) - \hat{x}(t))(x(t) - \hat{x}(t))^T\} \end{aligned}$$

- Assume R is diagonal, expand out gain:

$$L(t) = P(t)C^T \begin{bmatrix} R_{11}^{-1} & & \\ & \ddots & \\ & & R_{nn}^{-1} \end{bmatrix}$$

R_{11} large ⇒ smaller effect

P small ⇒ don't rely on sensors (state is accurate)

$$\dot{P} = \underbrace{AP + PA^T}_{} - PC^T R^{-1}(t) CP + FQ(t)F^T$$

Process disturbances

R small decreases uncertainty

P evolves according to nominal dynamics

- Steady state (ARE): optimal balance of dynamics and uncertainty

Variation #2: Extended Kalman Filter (EKF)

Consider a *nonlinear* system

$$\dot{x} = f(x, u, v) \qquad x \in \mathbb{R}^n, u \in \mathbb{R}^m$$

$$y = Cx + w \qquad v, w \text{ Gaussian white noise processes with covariance matrices } Q \text{ and } R.$$

Form estimator using nonlinear model + linear feedback

$$\dot{\hat{x}} = f(\hat{x}, u, 0) + L(y - C\hat{x})$$

Compute estimator gain based on linearization at current estimated state:

$$\dot{\hat{x}} = f(\hat{x}, u, 0) + L(y - C\hat{x}) \qquad L = PC^T R^{-1}$$

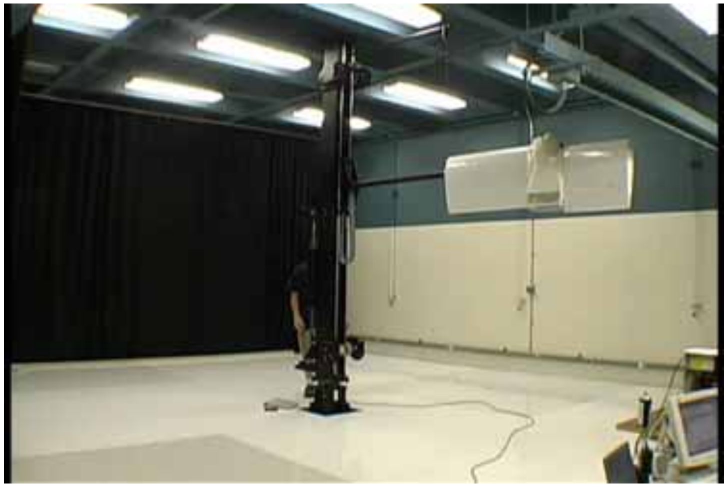
$$\dot{P} = (\tilde{A} - LC)P + P(\tilde{A} - LC)^T + \tilde{F}Q\tilde{F}^T + LRL^T \qquad P(t_0) = E\{x(t_0)x^T(t_0)\}$$

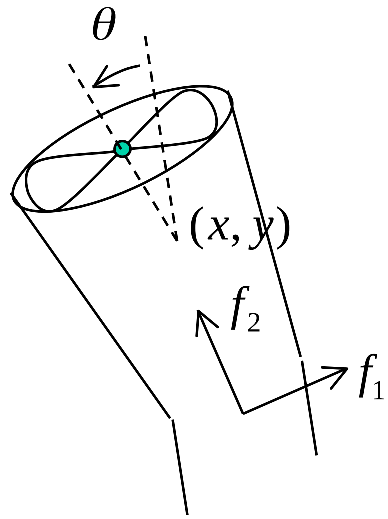
$$\tilde{A} = \left.\frac{\partial F}{\partial e}\right|_{(0,\hat{x},u,0)} = \left.\frac{\partial f}{\partial x}\right|_{(\hat{x},u,0)}$$

$$\tilde{F} = \left.\frac{\partial F}{\partial v}\right|_{(0,\hat{x},u,0)} = \left.\frac{\partial f}{\partial v}\right|_{(\hat{x},u,0)}$$

- Little formal theory, but works very well as long as estimated state is close
- Very important for tracking problems (might operate far from equilibrium)

Example: Ducted Fan





Estimation:

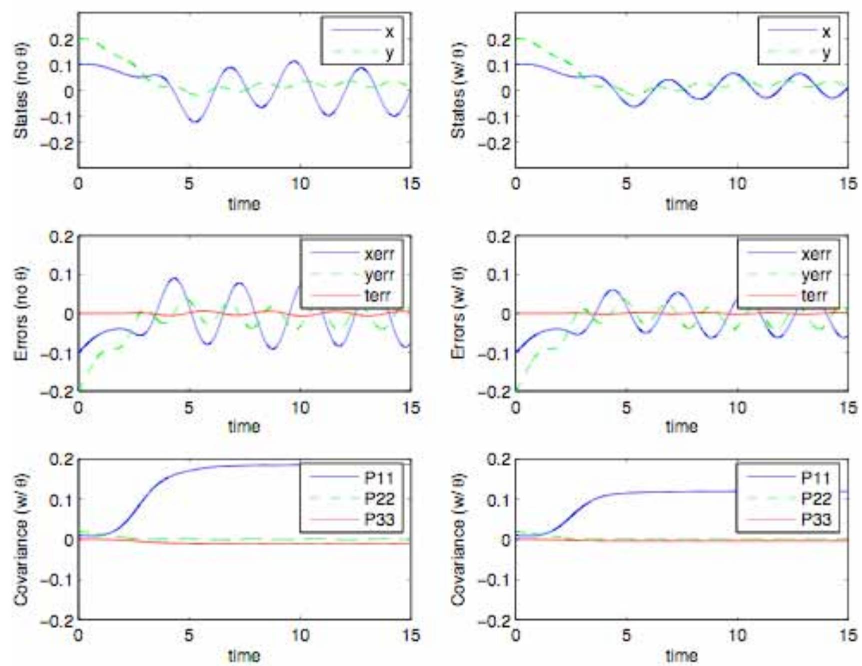
- Given the xy position of the fan and the inputs (f_1, f_2), determine the full state of the system:

$$x, y, \theta, \dot{x}, \dot{y}, \dot{\theta}$$

Equations of motion

$$m\ddot{x} = f_1 \cos\theta - f_2 \sin\theta - c_{d,x}(\theta, \dot{x})$$
$$m\ddot{y} = f_1 \sin\theta + f_2 \cos\theta - mg - c_{d,y}(\theta, \dot{y})$$
$$J\ddot{\theta} = r f_1 - mgl \sin\theta - c_{d,\theta}(\theta, \dot{\theta})$$

Estimator design: see dfan_kf.m, pvtol.m



Variation #3: Parameter Estimation

Suppose dynamics depend on unknown parameter ξ

$$\dot{x} = A(\xi)x + B(\xi)u + Fv \qquad \xi \in \mathbb{R}^p$$
$$y = C(\xi)x + w$$

Rewrite dynamics using added state ξ

$$\dot{x} = A(\xi)x + B(\xi)u + Fv$$
$$\dot{\xi} = 0$$

Now use extended Kalman filter to estimate state and parameter:

$$\frac{d}{dt}\begin{bmatrix} x \\ \xi \end{bmatrix} = \overbrace{\begin{bmatrix} A(\xi) & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ \xi \end{bmatrix} + \begin{bmatrix} B(\xi) \\ 0 \end{bmatrix} u}^{f(\left[\begin{smallmatrix} x \\ \xi \end{smallmatrix}\right], u, v)} + \begin{bmatrix} F \\ 0 \end{bmatrix} v$$
$$y = \underbrace{C(\xi)x + w}_{h(\left[\begin{smallmatrix} x \\ \xi \end{smallmatrix}\right], w)}$$

Variation #4: Discrete Time (ala Wikipedia)

Discrete-time dynamical system:

$$x_k = F x_{k-1} + B u_k + w_k \qquad x \in \mathbb{R}^n, u \in \mathbb{R}^m$$

$$z_k = H x_k + v_k \qquad w, v \text{ Gaussian white noise sequences w/ covariance matrices } Q \text{ and } R.$$

Kalman filter:

$$\hat{x}_{k|k-1} = F_k \hat{x}_{k-1|k-1} + B_k u_k$$

$$P_{k|k-1} = F_k P_{k-1|k-1} F_k^T + Q_k$$

← Predict

↙ Update

$$\tilde{y}_k = z_k - H_k \hat{x}_{k|k-1}$$ innovation or measurement residual

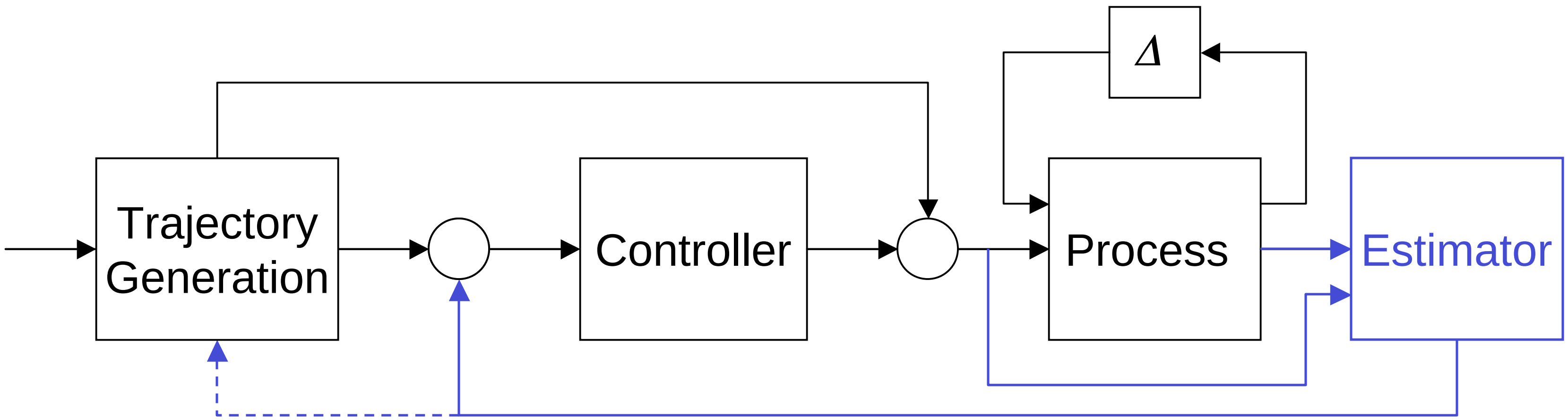
$$S_k = H_k P_{k|k-1} H_k^T + R_k$$ innovation (or residual) covariance

$$K_k = P_{k|k-1} H_k^T S_k^{-1}$$ Kalman gain

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k \tilde{y}_k$$ updated state estimate

$$P_{k|k} = (I - K_k H_k) P_{k|k-1}$$ updated estimate covariance

Course Overview



Estimation and Kalman filtering

- Given measurement of process input and output, predict current state
- Use stochastic description of noise and disturbance processes
- Optimal (Kalman) estimator: minimize covariance of the error
- Many variants: nonlinear (EKF), parameter estimation, discrete time
- HW 4: design several estimators using (continuous time) Kalman filter

Next

- Discuss *robust performance* of feedback systems (following DFT)