CDS 110b: Lecture 5-2
Kalman Filtering

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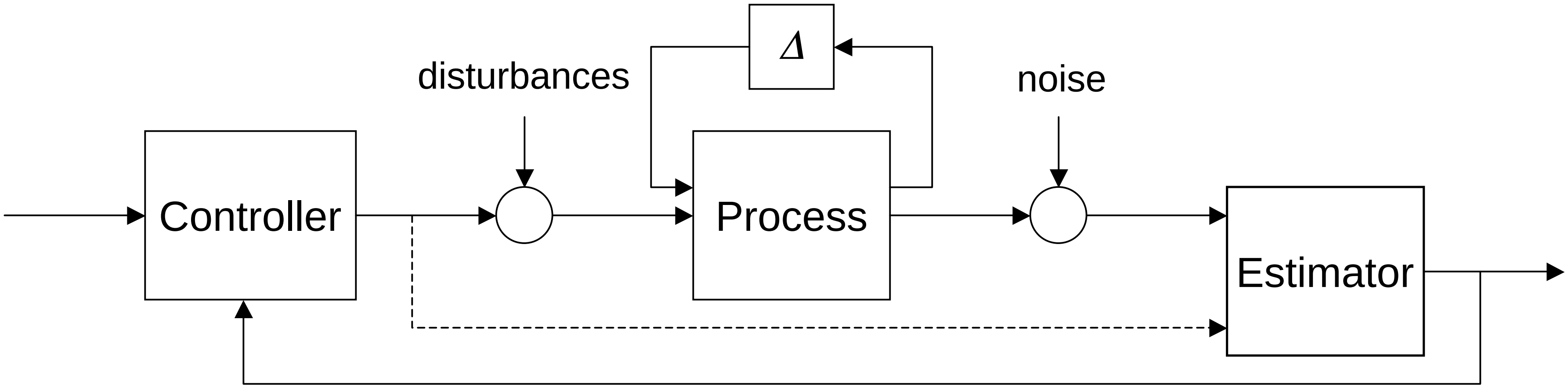
Goals:

- Give state space computations for stochastic system response
- Pose and describe the solution to the optimal estimation problem

Reading:

- Friedland, Chapter 11

The State Estimation Problem



Problem Setup

- Given a dynamical system with noise and uncertainty, estimate the state

$$\dot{x} = Ax + Bu + Fv$$
$$y = Cx + Du + Gw$$

$\longrightarrow$

$$\dot{\hat{x}} = \alpha(\hat{x}, y, u) \quad \longleftarrow \text{estimator}$$
$$\lim_{t \to \infty} E(x - \hat{x}) = 0$$

expected value

- $\hat{x}$ is called the *estimate* of x

Remarks

- Several sources of uncertainty: noise, disturbances, process, initial condition
- Uncertainties are unknown, except through their effect on measured output
- First question: when is this even *possible?*

Stochastic Response: State Space Computations

$$p_v(x) = \frac{1}{\sqrt{2\pi Q_v}} e^{-\frac{x^2}{2Q_v}}$$
$$R_v(\tau) = Q_v \delta(\tau)$$

v → $\dot{x} = Ax + Fv$, $y = Cx$ → y

$$p_y(x) = C \frac{1}{\sqrt{2\pi R_x(0)}} e^{-\frac{x^2}{2R_x(0)}}$$
$$R_y(\tau) = C R_x(\tau) C^T$$

Write solution of linear system in terms of *state transition matrix,* $\Phi : R^n \times R^n \to R^n$

$$\dot{x} = Ax + Fv \qquad \Phi(t, t_0) = e^{A(t-t_0)}$$

$$x(t) = \Phi(t, t_0) x(t_0) + \int_{t_0}^{t} \Phi(t, \lambda) F v(\lambda) d\lambda$$

Claim Let v be white noise with $E\{v(\lambda)v^T(\xi)\} = Q_v \delta(\lambda - \xi)$. Then the correlation matrix for x is given by

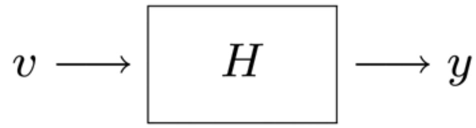
$$R_x(t, s) = P(t) \Phi^T(s, t) \qquad \text{where} \qquad \begin{aligned} \dot{P}(t) &= AP + PA^T + FQ_vF \\ P(0) &= E\{x(0)x^T(0)\}. \end{aligned}$$

Stationary case (steady state):

$$R_x(\tau) = Pe^{-A\tau} \qquad \text{where} \qquad AP + PA^T + FQ_vF^T = 0 \qquad P > 0$$

Summary: Stochastic Response

$$p(v) = \frac{1}{\sqrt{2\pi Q_v}} e^{-\frac{x^2}{2Q_v}}$$
$$S_v(\omega) = Q_v$$



$$p(y) = \frac{1}{\sqrt{2\pi R_y}} e^{-\frac{x^2}{2R_y}}$$
$$S_y(\omega) = H(-j\omega) Q_v H(j\omega)$$

$$\rho_v(\tau) = Q_v \delta(\tau)$$

$$\dot{x} = Ax + Fv$$
$$y = Cx$$

$$\rho_y(\tau) = R_y(\tau) = CPe^{-A\tau}C^T$$
$$AP + PA^T + FQ_vF^T = 0$$

Remarks

- Both v and y are *random processes* (not signals)
- Transformations describe how the statistics of the process are mapped through a linear system
- Computations can be done either in frequency domain or time domain (state space)
- Can also work out equations for discrete time systems (useful in signal processing and a bit easier to work with)

Optimal Estimation

System description

$$\dot{x} = Ax + Bu + Fv \qquad E\{v(s)v^T(t)\} = Q(t)\delta(t-s)$$
$$\dot{y} = Cx + w \qquad E\{w(s)w^T(t)\} = R(t)\delta(t-s)$$

- Disturbances and noise are multi-variable Gaussians with covariance Q, R

Problem statement: Find the estimate that minimizes the mean square error $E\{(x(t)-\hat{x}(t))(x(t)-\hat{x}(t))^T\}$ given $\{u(t), y(t): 0 \le t \le T\}$.

Proposition $\hat{x}(t) = E\{x(t) | y(\tau), \tau \le t\}$

- Optimal estimate is just the expectation of the random process x given the *constraint* of the observed output.
- This is the way Kalman originally formulated the problem.
- Can think of this as a *least squares* problem: given all previous $y(t)$, find the estimate $\hat{x}(t)$ that satisfies the dynamics and minimizes the square error with the measured data.

Kalman-Bucy Filter

Theorem 1 (Kalman-Bucy, 1961). *The optimal estimator has the form of a linear observer*

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$$

where $L(t) = P(t)C^T R^{-1}$ & $P(t) = E\{(x(t)-\hat{x}(t))(x(t)-\hat{x}(t))^T\}$ *satisfies*

$$\dot{P} = AP + PA^T - PC^T R^{-1}(t)CP + FQ(t)F^T$$
$$P(0) = E\{x(0)x^T(0)\}$$

Proof. (sketch) The error dynamics are given by

$$\dot{e} = (A - LC)e + \xi \qquad \xi = Fv - Lw \qquad R_\xi = FQF^T + LRL^T$$

The covariance matrix $P_e = P$ for this process satisfies

$$\dot{P} = (A - LC)P + P(A - LC)^T + FQF^T + LRL^T.$$

We need to find L such that $P(t)$ is as small as possible. Can show that the L that acheives this is given by

$$RL^T = CP \qquad \Longrightarrow \qquad L = PC^T R^{-1}$$

Kalman-Bucy Filter

1. The Kalman filter has the form of a *recursive* filter: given $P(t) = E\{e(t)e^T(t)\}$ at time t, can compute how the estimate and covariance *change*. Don't need to keep track of old values of the output.

2. The Kalman filter gives the estimate $\hat{x}(t)$ *and* the covariance $P_e(t)$ $\Longrightarrow$ you can see how well the error is converging.

3. If the noise is stationary (Q, R constant) and *if* $\dot{P}$ is stable, then the observer gain is constant:

$$L = PC^T R^{-1} \qquad AP + PA^T - PC^T R^{-1} CP + FQF^T$$

   This is the problem solved by the `lqe` command in MATLAB.

4. Kalman filter extracts max possible information about output data

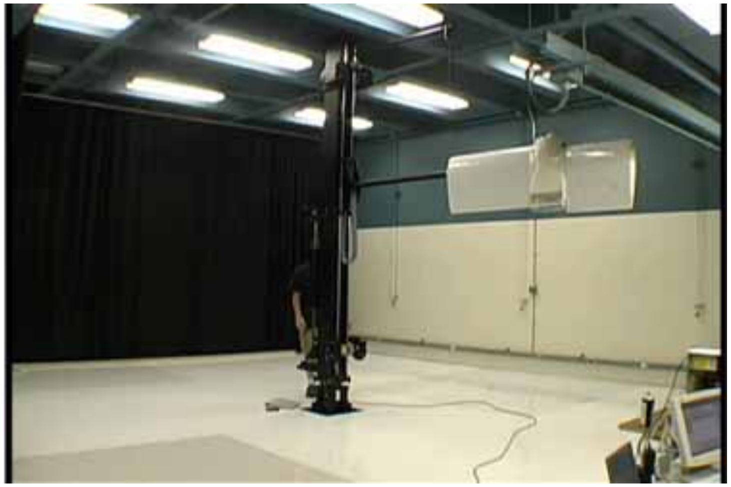
$$r = y - C\hat{x} = \text{residual or } \textit{innovations} \text{ process}$$

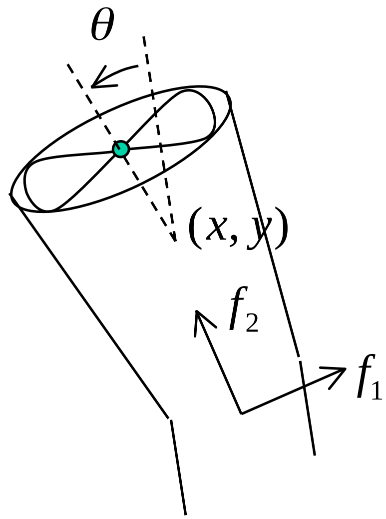
   Can show that for the Kalman filter, the correlation matrix is

$$R_r(t, s) = W(t)\delta(t - s) \qquad \Longrightarrow \qquad \text{white noise}$$

   So the output error has *no* remaining dynamic information content

Example: Ducted Fan





Estimation:

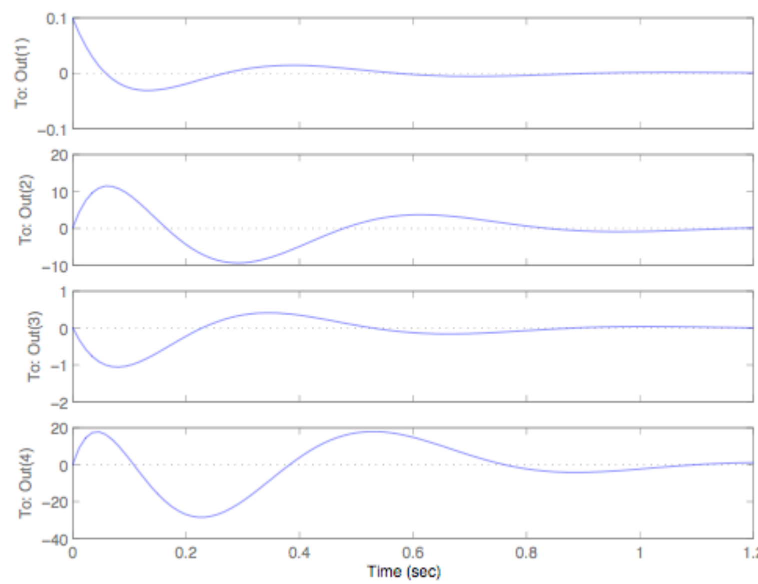
- Given the xy position of the fan and the inputs (f_1, f_2), determine the full state of the system:

$$x, y, \theta, \dot{x}, \dot{y}, \dot{\theta}$$

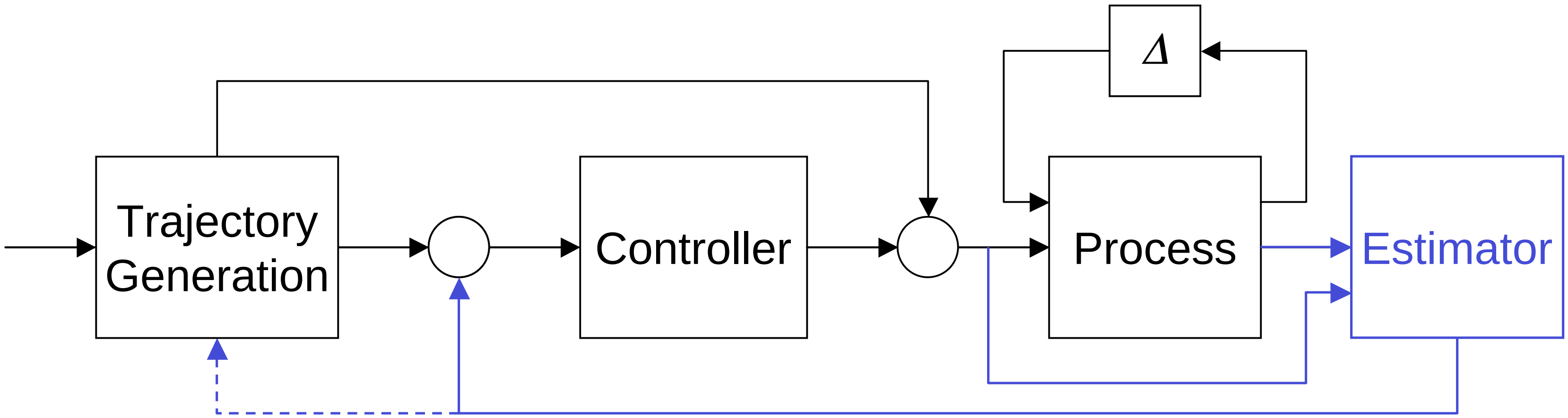
Equations of motion

$$m\ddot{x} = f_1 \cos\theta - f_2 \sin\theta - c_{d,x}(\theta, \dot{x})$$
$$m\ddot{y} = f_1 \sin\theta + f_2 \cos\theta - mg - c_{d,y}(\theta, \dot{y})$$
$$J\ddot{\theta} = r f_1 - mgl \sin\theta - c_{d,\theta}(\theta, \dot{\theta})$$

Estimator design: see obs_dfan.m



Separation Principle



Stochastic control problem: find $C(s)$ to minimize

$$J = E\left\{\int_0^\infty \left[(y-r)^T Q (y-r)^T + u^T R u\right]\, dt\right\}$$

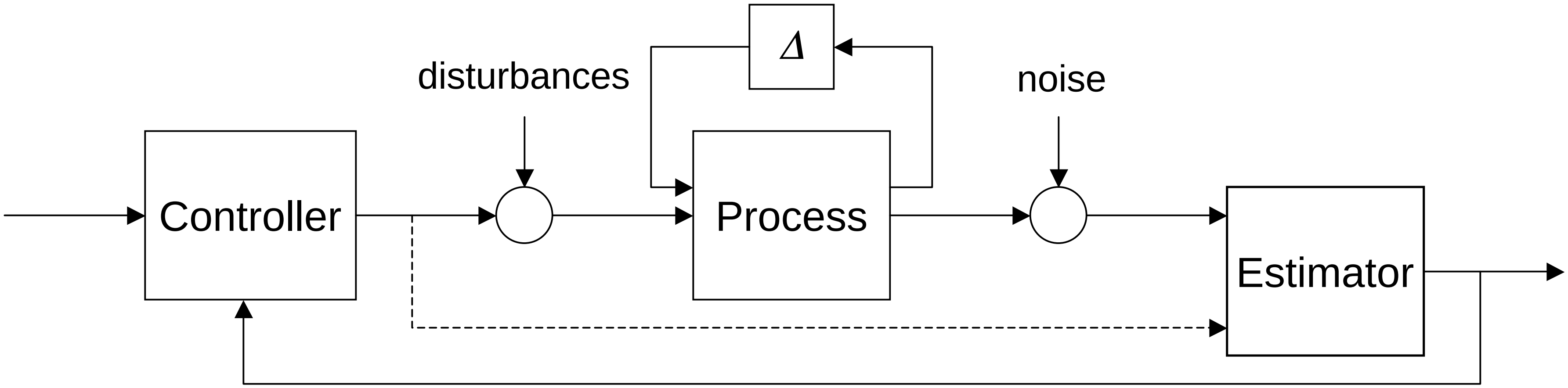
Assume for simplicity that $r = 0$ (otherwise, translate state accordingly).

Theorem 1. *The optimal controller has the form*

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$$
$$u = K(\hat{x} - x_d)$$

where L is the optimal observer gain ignoring the controller and K is the optimal controller gain ignoring the noise.

Summary: Observers and State Estimation



Use stochastic systems framework

- Model disturbances and noise as random processes; characterize by first and second order statistics (mean, variance; correlation)

Kalman filter = optimal estimator

- Assumes Gaussian white noise; creates best estimate given data
- Implemented as a recursive filter => keep track of estimate + covariance
- *Extremely* useful in a broad variety of applications (more next week)