Goals:
- Give state space computations for stochastic system response
- Pose and describe the solution to the optimal estimation problem

Reading:
- Friedland, Chapter 11
Problem Setup

- Given a dynamical system with noise and uncertainty, estimate the state

\[
\dot{x} = Ax + Bu + Fv \\
y = Cx + Du + Gw \\
\hat{x} = \alpha(\hat{x}, y, u)
\]

- \(\hat{x}\) is called the estimate of \(x\)

Remarks

- Several sources of uncertainty: noise, disturbances, process, initial condition
- Uncertainties are unknown, except through their effect on measured output
- First question: when is this even possible?
Stochastic Response: State Space Computations

\[ p_v(x) = \frac{1}{\sqrt{2\pi Q_v}} e^{-\frac{x^2}{2Q_v}} \]

\[ R_v(\tau) = Q_v \delta(\tau) \]

\[
\begin{align*}
\dot{x} &= A x + F v \\
y &= C x
\end{align*}
\]

\[ p_y(x) = C \frac{1}{\sqrt{2\pi R_x(0)}} e^{-\frac{x^2}{2R_x(0)}} \]

\[ R_y(\tau) = CR_x(\tau)C^T \]

Write solution of linear system in terms of state transition matrix, \( \Phi: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n \)

\[ \dot{x} = Ax + Fv \quad \Phi(t, t_0) = e^{A(t-t_0)} \]

\[ x(t) = \Phi(t, t_0)x(t_0) + \int_{t_0}^{t} \Phi(t, \lambda)Fv(\lambda)d\lambda \]

Claim Let \( v \) be white noise with \( E\{v(\lambda)v^T(\xi)\} = Q_v \delta(\lambda - \xi) \). Then the correlation matrix for \( x \) is given by

\[ R_x(t, s) = P(t)\Phi^T(s, t) \quad \text{where} \quad \dot{P}(t) = AP + PA^T + FQ_vF \]

\[ P(0) = E\{x(0)x^T(0)\}. \]

Stationary case (steady state):

\[ R_x(\tau) = Pe^{-A\tau} \quad \text{where} \quad AP + PA^T + FQ_vF^T = 0 \quad P > 0 \]
Summary: Stochastic Response

\[ p(v) = \frac{1}{\sqrt{2\pi Q_v}} e^{-\frac{x^2}{2Q_v}} \]
\[ S_v(\omega) = Q_v \]
\[ \rho_v(\tau) = Q_v \delta(\tau) \]
\[ v \rightarrow H \rightarrow y \]

\[ p(y) = \frac{1}{\sqrt{2\pi R_y}} e^{-\frac{x^2}{2R_y}} \]
\[ S_y(\omega) = H(-j\omega)Q_vH(j\omega) \]
\[ \rho_y(\tau) = R_y(\tau) = CPe^{-A\tau}C^T \]
\[ AP + PAT^T + FQ_vF^T = 0 \]

Remarks

- Both \( v \) and \( y \) are random processes (not signals)
- Transformations describe how the statistics of the process are mapped through a linear system
- Computations can be done either in frequency domain or time domain (state space)
- Can also work out equations for discrete time systems (useful in signal processing and a bit easier to work with)
Optimal Estimation

System description

\[
\begin{align*}
\dot{x} &= Ax + Bu + Fv \\
\dot{y} &= Cx + w
\end{align*}
\]

\[
E\{v(s)v^T(t)\} = Q(t)\delta(t - s) \\
E\{w(s)w^T(t)\} = R(t)\delta(t - s)
\]

• Disturbances and noise are multi-variable Gaussians with covariance \(Q, R\)

Problem statement: Find the estimate that minimizes the mean square error
\[
E\{(x(t) - \hat{x}(t))(x(t) - \hat{x}(t))^T\}
\]
given \(u(t), y(t): 0 \leq t \leq T\).

Proposition \(\hat{x}(t) = E\{x(t)|y(\tau), \tau \leq t\}\)

• Optimal estimate is just the expectation of the random process \(x\) given the constraint of the observed output.
• This is the way Kalman originally formulated the problem.
• Can think of this as a least squares problem: given all previous \(y(t)\), find the estimate \(\hat{x}(t)\) that satisfies the dynamics and minimizes the square error with the measured data.
Kalman-Bucy Filter

Theorem 1 (Kalman-Bucy, 1961). The optimal estimator has the form of a linear observer

\[ \dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x}) \]

where \( L(t) = P(t)C^TR^{-1} \) & \( P(t) = E\{(x(t) - \hat{x}(t))(x(t) - \hat{x}(t))^T\} \) satisfies

\[ \dot{P} = AP + PA^T - PC^TR^{-1}(t)CP + FQ(t)F^T \]

\[ P(0) = E\{x(0)x^T(0)\} \]

Proof. (sketch) The error dynamics are given by

\[ \dot{e} = (A - LC)e + \xi \quad \xi = Fv - Lw \quad R_\xi = FQF^T + LRL^T \]

The covariance matrix \( P_e = P \) for this process satisfies

\[ \dot{P} = (A - LC)P + P(A - LC)^T + FQF^T + LRL^T. \]

We need to find \( L \) such that \( P(t) \) is as small as possible. Can show that the \( L \) that achieves this is given by

\[ RL^T = CP \quad \implies \quad L = PC^TR^{-1} \]
Kalman-Bucy Filter

1. The Kalman filter has the form of a recursive filter: given $P(t) = E\{e(t)e^T(t)\}$ at time $t$, can compute how the estimate and covariance change. Don’t need to keep track of old values of the output.

2. The Kalman filter gives the estimate $\hat{x}(t)$ and the covariance $P_e(t)$ $\implies$ you can see how well the error is converging.

3. If the noise is stationary ($Q, R$ constant) and if $\dot{P}$ is stable, then the observer gain is constant:

$$L = P C^T R^{-1} \quad A P + P A^T - P C^T R^{-1} C P + F Q F^T$$

This is the problem solved by the $\text{lqe}$ command in MATLAB.

4. Kalman filter extracts max possible information about output data

$$r = y - C \hat{x} = \text{residual or innovations}$$

Can show that for the Kalman filter, the correlation matrix is

$$R_r(t, s) = W(t) \delta(t - s) \implies \text{white noise}$$

So the output error has no remaining dynamic information content
Example: Ducted Fan

Equations of motion

\begin{align*}
    m\ddot{x} &= f_1 \cos \theta - f_2 \sin \theta - c_{d,x}(\theta, \dot{x}) \\
    m\ddot{y} &= f_1 \sin \theta + f_2 \cos \theta - mg - c_{d,y}(\theta, \dot{y}) \\
    J\ddot{\theta} &= rf_1 - mgl \sin \theta - c_{d,\theta}(\theta, \dot{\theta})
\end{align*}

Estimator design: see obs_dfan.m

Estimation:

- Given the \( xy \) position of the fan and the inputs \((f_1, f_2)\), determine the full state of the system:
  \[ x, y, \theta, \dot{x}, \dot{y}, \dot{\theta} \]
Stochastic control problem: find $C(s)$ to minimize

$$J = E \left\{ \int_{0}^{\infty} \left[ (y - r)^T Q(y - r)^T + u^T R u \right] dt \right\}$$

Assume for simplicity that $r = 0$ (otherwise, translate state accordingly).

**Theorem 1.** The optimal controller has the form

$$\hat{x} = A\hat{x} + Bu + L(y - C\hat{x})$$
$$u = K(\hat{x} - x_d)$$

where $L$ is the optimal observer gain ignoring the controller and $K$ is the optimal controller gain ignoring the noise.
Summary: Observers and State Estimation

Use stochastic systems framework
- Model disturbances and noise as random processes; characterize by first and second order statistics (mean, variance; correlation)

Kalman filter = optimal estimator
- Assumes Gaussian white noise; creates best estimate given data
- Implemented as a recursive filter => keep track of estimate + covariance
- Extremely useful in a broad variety of applications (more next week)