CDS 110b: Lecture 1-2
Introduction to Optimal Control

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Goals:

- Introduce the problem of *optimal* control as method of trajectory generation
- State the maximum principle and give examples of its usage

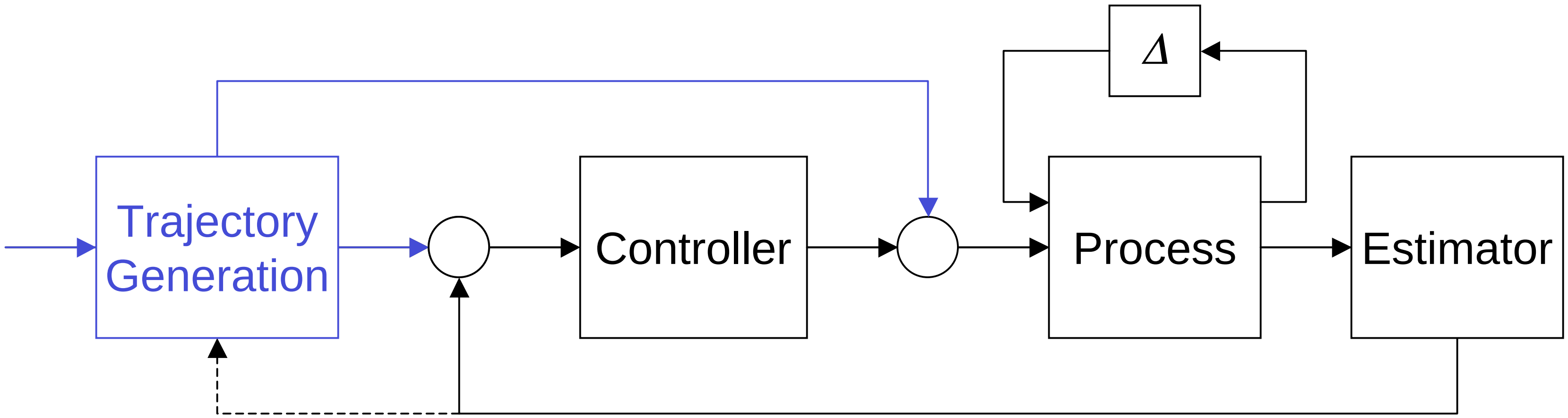
Reading:

- Lewis and Syrmos, Section 3.1-3.2 (available on web page)
- RMM course notes (available on web page)

Homework #1

- Apply the maximum principle to some simple problems
- Due Wed, 11 Jan by 5 pm, in box outside 109 Steele

Trajectory Generation



System Dynamics: $\dot{x} = f(x, u)$

Trajectory Generation: $r \rightarrow (x_d, u_d)$

Tracking Controller: $u = u_d + K(x - x_d)$

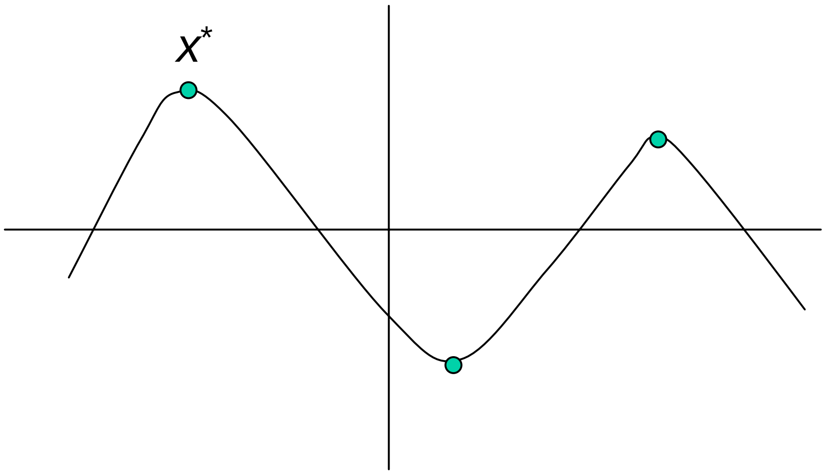
Approach: optimal control

- Find trajectory that minimizes desired cost function and satisfies the (nonlinear) dynamics
- Use linear state feedback (K) that minimizes quadratic criteria

Modern variation: receding horizon control

- Resolve trajectory generation in real-time, based on current state

Review: Optimization



Optimization of functions: given $F: R^n \to R$, find $x^* \in R^n$ such that $F(x^*) \geq F(x)$ for all $x \in R^n$

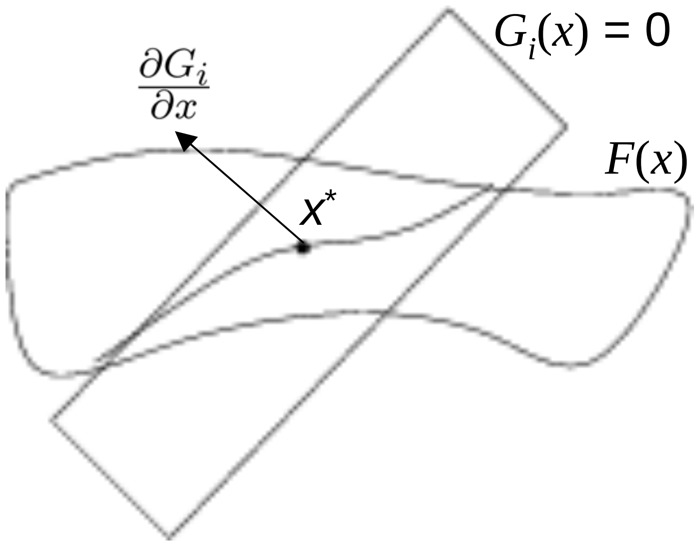
Necessary conditions:

$$\frac{\partial F}{\partial x}(x^*) = 0$$

Remarks:

- Basic idea: if gradient is non-zero, move in that direction to increase cost
- Conditions are *not* sufficient conditions: all marked points satisfy conditions, only one is (global) maximum
- To find the *minimum* of a function *F*, find the maximum of the function *-F* $\Rightarrow$ conditions are unchanged (so we will switch back and forth…)
- MATLAB: `X = FMINSEARCH(FUN,X0) starts at X0 and attempts to find a local minimizer X of the function FUN`

Constrained Optimization



Optimization with *constraints*: given cost function $F: R^n \to R$ and constraints $G_i: R^n \to R$, $i = 1, \ldots, k$, find $x^* \in R^n$ such that $G_i(x^*) = 0$ (satisfies constraints) and $F(x^*) \geq F(x)$ for all x such that $G_i(x) = 0$.

Necessary conditions

$$\frac{\partial F}{\partial x}(x^*) + \sum_{i=1}^{k} \lambda_i \frac{\partial G_i}{\partial x}(x^*) = 0$$

Lagrange multipliers: λ_i

- Geometric interpretation: free variables that "cancel" the gradient of *F* in directions normal to the constraint
- Algebraic interpretation: minimize the function $\widetilde{F} = F + \sum \lambda_i G_i$ with respect to x, leaving λ free. Variables λ can take any value $\Rightarrow$ need to choose x such that $G(x) = 0$ (otherwise, we can choose λ to generate a large cost)

MATLAB: use Optimization Toolbox

Optimal Control

Problem Statement: Given nonlinear system

$$\dot{x} = f(x, u) \qquad x \in \mathbb{R}^n,\ u \in \mathbb{R}^m$$

Find trajectory (x^*, u^*) that satisfies the dynamics and minimizes cost

$$\min_u \underbrace{\int_0^T L(x, u)\, dt}_{\text{Integrated cost}} + \underbrace{V(x(T), u(T))}_{\text{Terminal cost}}$$

Remarks

- Usually assume we are trying to take system to the origin ($x = 0$); easy to generalize by rewriting dynamics in terms of write $e = x - r$
- Integrated cost $L(x,u)$ is used to tradeoff state error (x) from input cost (u)
- Typical cost function: *quadratic cost*

$$J = \frac{1}{2}\int_0^T \left(x^T Q x + u^T R u\right) dt + x^T(T) P_1 x(T)$$

Pontryagin's Maximum Principle

System:

$$\dot{x} = f(x,u) \quad x = \mathbb{R}^n$$
$$x(0) \text{ given} \quad u \in \Omega \subset \mathbb{R}^p$$

Cost:

$$J = \int_0^T L(x,u)\,dt + V(x(T))$$
$$\psi(x(T)) = 0$$

← Terminal constraint (optional)

Hamiltonian: $H = L + \lambda^T f = L + \sum \lambda_i f_i$

Theorem (Pontryagin): If (x^*, u^*) is optimal, then there exists $\lambda^*(t)$ and ν^* such that

$$\dot{x}_i = \frac{\partial H}{\partial \lambda_i} \qquad -\dot{\lambda}_i = \frac{\partial H}{\partial x_i}$$

$$\left.\begin{aligned} &x(0) \text{ given}, \ \psi(x(T)) = 0 \\ &\lambda(T) = \frac{\partial V}{\partial x}(x(T)) + \frac{\partial \psi^T}{\partial x}\nu \end{aligned}\right\}$$

Boundary conditions ($2n$ total)

and

$$H(x^*(t), u^*(t), \lambda^*(t)) \leq H(x^*(t), u, \lambda^*(t)) \quad \forall \quad u \in \Omega$$

Remark: this is a *very* general (and useful) set of conditions

Using the Maximum Principle

1. Formulate problem in standard form:

$$\dot{x} = f(x,u) \quad x = \mathbb{R}^n$$
$$x(0) \text{ given} \quad u \in \Omega \subset \mathbb{R}^p$$

$$J = \int_0^T L(x,u)\,dt + V(x(T))$$
$$\psi(x(T)) = 0$$

2. Construct Hamiltonian: $H = L + \lambda^T f = L + \sum \lambda_i f_i$

3. Compute necessary conditions:

$$\dot{x}_i = \frac{\partial H}{\partial \lambda_i} \qquad -\dot{\lambda}_i = \frac{\partial H}{\partial x_i}$$

$$x(0) \text{ given}, \ \psi(x(T)) = 0$$
$$\lambda(T) = \frac{\partial V}{\partial x}(x(T)) + \frac{\partial \psi^T}{\partial x}\nu$$

4. Find the optimal input

$$u = \arg\min \ H(x^*(t), u, \lambda^*(t))$$

5. Solve for the optimal trajectory:

- Substitute optimal input into necessary conditions and solved boundary value problem. In general, this is hard to do in closed form
- Good news: can convert this to a computational problem

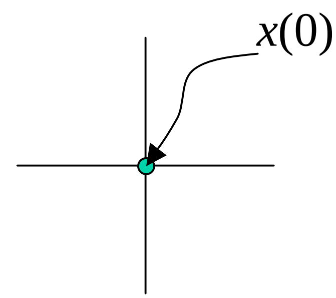
Example: Bang-Bang Control

1. Problem formulation: move to origin in minimum amount of time

$$\dot{x} = Ax + Bu \qquad |u| \leq 1$$ ← Linear dynamics, bounded input

$$J = \int_0^T 1\, dt$$ ← Minimum time (make T small)

$$\psi(x(T)) = x(T)$$ ← Terminal constraint



2. Construct Hamiltonian: $H = L + \lambda^T f(x, u) = 1 + (\lambda^T A)x + (\lambda^T B)u$

3, 4. Compute necessary conditions and optimal input:

$$\dot{x} = \frac{\partial H}{\partial \lambda} = Ax + Bu$$ ← Returns a copy of the dynamics

$$-\dot{\lambda} = \frac{\partial H}{\partial x} = A^T \lambda$$ ← Evolution of Lagrange multipliers (costates)

$$u = \text{arg min } H = -\text{sgn}(\lambda^T B)$$ ← Optimal input

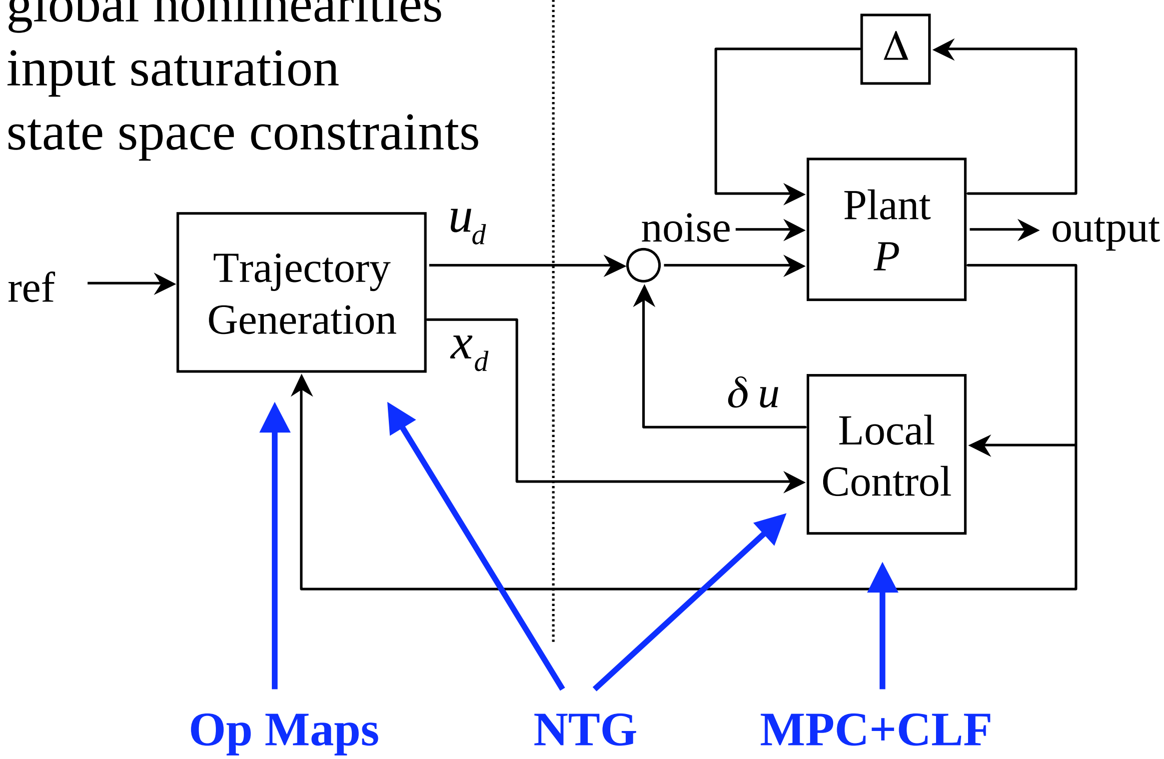
Remarks:

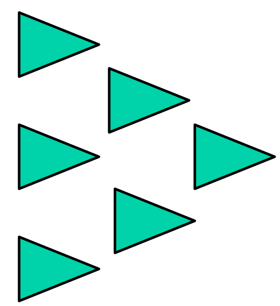
- Form of the solution is easy: apply max (or min) input at all times
- Finding *actual* trajectory is hard: need to search over switching times

Application: Ducted Fan (flight control)

Nonlinear design
- global nonlinearities
- input saturation
- state space constraints

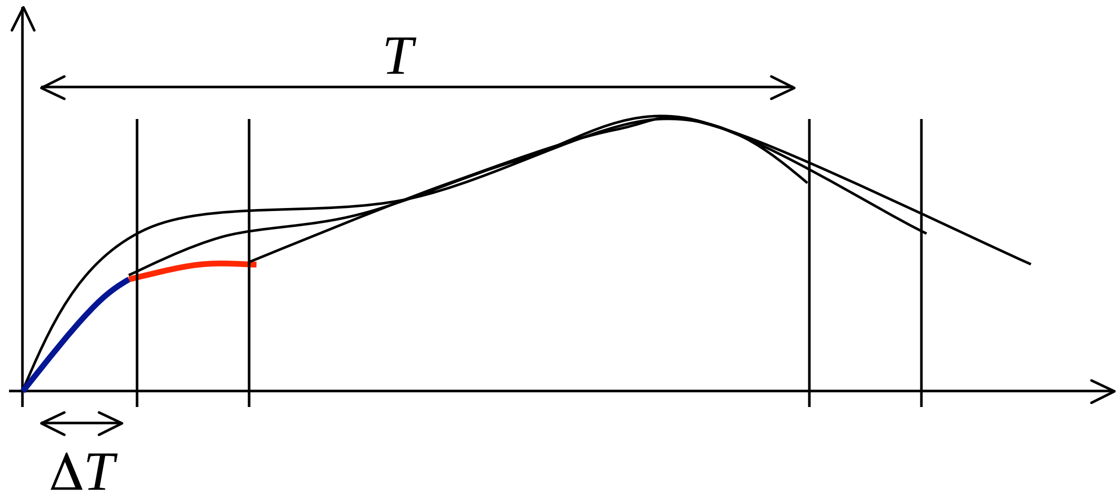
Local design





- Use *real-time* trajectory generation to construct (suboptimal) feasible trajectories
- Use model predictive control for *reconfigurable* tracking & robust performance
- Extension to multi-vehicle systems performing cooperative tasks

Approach: Receding Horizon Optimization



$$u_{[t,t+\Delta T]} = \arg\min \int_t^{t+T} L(x(\tau),u(\tau))\,d\tau + V(x(t+T))$$

$$x_0 = x(t) \qquad x_f = x_d(t+T)$$

$$\dot{x} = f(x,u) \qquad g(x,u) \le 0$$

Online control customization

- System: f(x,u)
- Constraints/environment: g(x,u)
- Mission: L(x,u)

Update in real-time to achieve *reconfigurable* operation

Real-Time Trajectory Generation / Optimization

$$\dot{x} = f(x,u)$$

Collocation

$$(x,u) = \sum \alpha_i \psi^i(t)$$

$$\dot{x}(t_i) = f(x(t_i), u(t_i))$$

Flatness

$$z = z(x, u, \dot{u}, \ldots, u^{(p)})$$

$$x = x(z, \dot{z}, \ldots, z^{(q)})$$

$$u = u(z, \dot{z}, \ldots, z^{(q)})$$

$$z = \sum \alpha_i \psi^i(t)$$

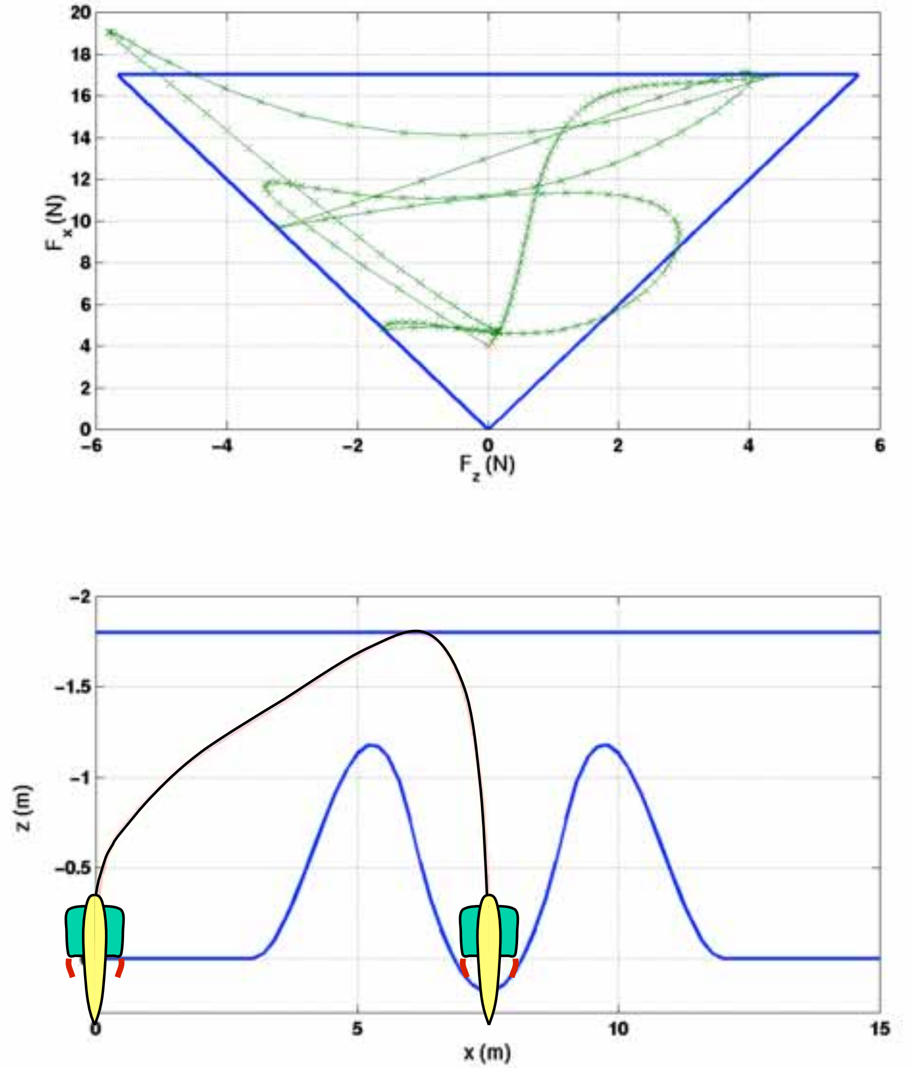
Quasi-collocation

$$y = h(x)$$

$$(x,u) = \Gamma(y, \dot{y}, \ldots, y^{(q)})$$

$$0 = \Phi(y, \dot{y}, \ldots, y^{(p)})$$

Ducted Fan Terrain Avoidance



Experimental Results: Caltech Ducted Fan

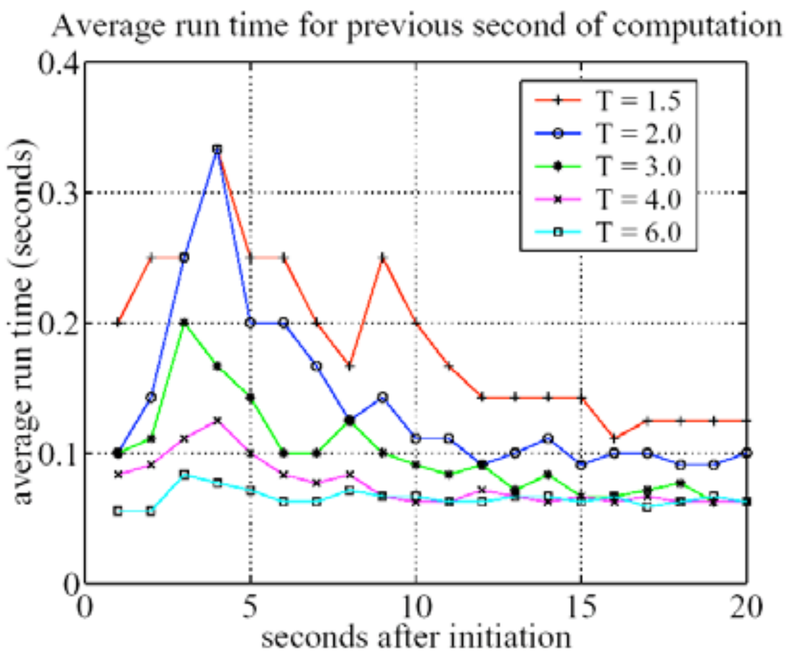
NTG + MPC + CLF

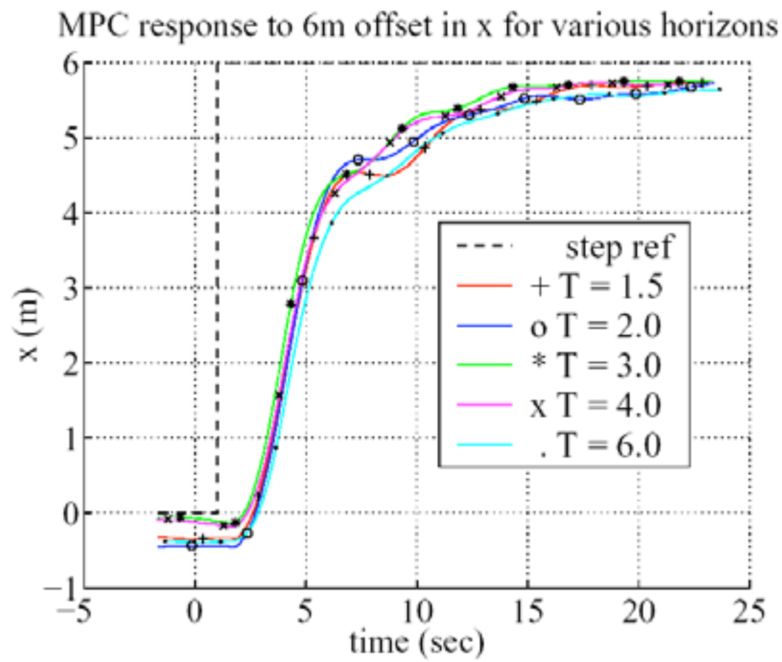
Pitch Control

dSPACE RTOS

Milam, Franz, Hauser, 2002 ACC
Dunbar, Milam, Murray, 2002 IFAC

Experiments: Caltech Ducted Fan

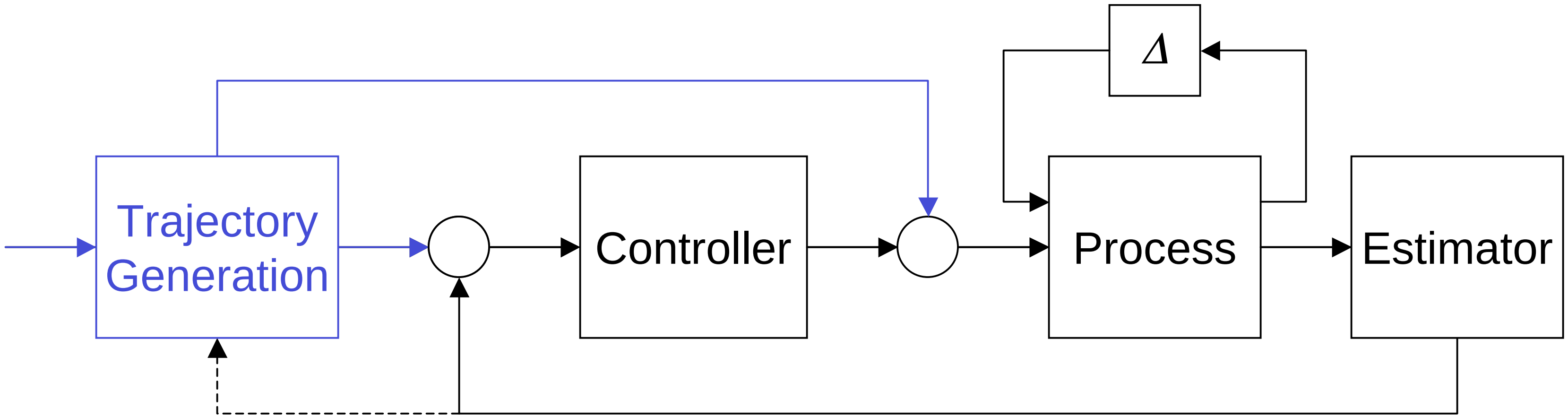




Real-Time MPC on Caltech Ducted Fan (Aug 01)

- NTG with quasi-flat outputs + *Lyapunov CLF*
- Average computation time of ~100 msec
- Inner (pitch) loop closed using local control law; MPC for position variables
- Inner/outer tradeoff: how much can be pushed into optimization

Summary



Optimal Control for Trajectory Generation

- Find feasible, optimal trajectories for a (nonlinear) control system
- Necessary conditions: Pontryagin Maximum Principle

Homework

- Work through several examples of maximum principle

Next week: application of optimal control to design state feedback

Note: NO CLASS MONDAY - next regular class on Wed, 11 Jan, 1:30 pm

Course Project information meeting: Friday, 2-3 pm, 125 Steele