CALIFORNIA INSTITUTE OF TECHNOLOGY
Control and Dynamical Systems

CDS 110
Problem Set #9

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Due: 13 Jan 03

Note: In the upper left hand corner of the first page of your homework set, please put the number of hours that you spent on this homework set (including reading).

Unless otherwise specified, you may use MATLAB to solve for pole placement gains.

1. The linearized equations of motion of a simple pendulum are given by

$$\ddot{\theta} + \omega^2 \theta = u$$

   (a) Write the equations of motion in state-space form.

   (b) (Without MATLAB) Design an estimator (observer) that reconstructs the state of the pendulum given measurements of $\dot{\theta}$. Assume ω=5 rad/sec, and pick the estimator roots to be at $s = -10 \pm 10j$.

   (c) Write the transfer function of the estimator between the measured value of $\dot{\theta}$ and the estimated value of θ.

   (d) (Without MATLAB) Design a controller (that is, determine the state feedback gain K) so that the roots of the closed-loop characteristic equation are at $s = -4 \pm 4j$.

2. Consider the system

$$\dot{x} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & -5 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix} u$$

   (a) For the observation $y = [1\ 0\ 0]x$, design an observer with eigenvalues at -10.

   (b) For the observation $y = [0\ 1\ 0]x$, try to design an observer with eigenvalues also at -10. What is the cause of the problem you encounter?

3. (Friedland 2.1, 3.6, 7.2) Consider the inverted pendulum on a cart driven by an electric motor. The equations of motion for the system are given by

$$\ddot{x} + \frac{k^2}{Mr^2R}\dot{x} + \frac{mg}{M}\theta = \frac{k}{MRr}e$$
$$\ddot{\theta} - \left(\frac{M+m}{Ml}\right) g\theta - \frac{k^2}{Mr^2Rl}\dot{x} = -\frac{k}{MRrl}e$$

   where k is the motor torque constant, R is the motor resistance, r is the ratio of the linear forces applied to the cart ($\tau = rf$), and e is the voltage applied to the motor. The following numerical data may be used:

$$m = 0.1 \text{ kg} \qquad M = 1.0 \text{ kg} \qquad l = 1.0 \text{ m} \qquad g = 9.8 \text{ m/s}^2$$
$$k = 1 \text{ V} \cdot \text{s} \qquad R = 100\ \Omega \qquad r = 0.02 \text{ m}$$

   An observer for the inverted pendulum on a motor-driven cart is to be designed using the measurement of the displacement of the cart ($y = x$).

(a) Find the matrices A, B, C, and D of the state-space characterization of the system.

(b) Determine the observer gain for which the observer poles lie in a fourth-order Butterworth pattern of radius 5, i.e., the characterstic equation is to be

$$\left(\frac{s}{5}\right)^4 + 2.613\left(\frac{s}{5}\right)^3 + (2+\sqrt{2})\left(\frac{s}{5}\right)^2 + 2.613\left(\frac{s}{5}\right)^1 + 1 = 0.$$

(c) Plot the response of the observer states to an initial state estimate error of $(1, 1, 1, 1)$.

4. (Friedland 6.1a, 8.4) A compensator based on a full-order observer is to be designed for the inverted pendulum on the motor-driven cart of the last problem.

   (a) Design a state space controller for the system that places the dominant poles (in a Butterworth configuration) at

   $$s = -4 \qquad \text{and} s = -2 \pm j2\sqrt{3}$$

   leaves the pole at $s = -25$ unchanged.

   (b) Using the regulator gains of part (a) and the observer gains of problem 3, determine the transfer function $D(s)$ of the compensator.

   (c) Assume a gain variation at the control input so that the loop transfer function is $L(s) = kD(s)H(s)$, where $H(s)$ is the transfer function of the plant. Find the range of k for which the closed-loop system is stable.

   (d) Plot the step response of the closed-loop system to a change in the desired x location of the cart. Compute the rise time, settling time, overshoot and steady state error for your controller.