CALIFORNIA INSTITUTE OF TECHNOLOGY
Control and Dynamical Systems

CDS 110
Problem Set #12

R. M. Murray
Winter 2003

Issued: 29 Jan 03
Due: 5 Feb 03

Note: In the upper left hand corner of the first page of your homework set, please put the number of hours that you spent on this homework set (including reading).

Unless otherwise specified, you may use MATLAB or Mathematica as long as you include a copy of the code used to generate your answer.

1. A random variable y is the sum of two independent normally (Gaussian) distributed random variables having means m_1, m_2 and variances σ_1^2, σ_2^2 respectively. Show that the probability density function for y is

$$p(y) = \frac{1}{2\pi\sigma_1\sigma_2} \int_{-\infty}^{\infty} \exp\left\{ -\frac{(y-x-m_1)^2}{2\sigma_1^2} - \frac{(x-m_2)^2}{2\sigma_2^2} \right\} dx$$

   and confirm that this is normal (Gaussian) with mean $m_1 + m_2$ and variance $\sigma_1^2 + \sigma_2^2$.

2. Find a constant matrix A and vectors F and C such that for

$$\dot{x} = Ax + Fw, \; y = Cx$$

   the power spectrum of y is given by

$$S(\omega) = \frac{1+\omega^2}{(1-7\omega^2)^2 + 1}$$

   Describe the sense in which your answer is unique.

3. (Friedland 11.1) A compensator based on a Kalman filter is to be designed for the instrument servo problem of HW #10, problem 2 (Friedland 9.6). Only the position error e is measured, so that

$$y = e + w$$

   where w is white noise with spectral density W. The only excitation noise present occurs at the control input, so that the angular velocity is

$$\dot{\omega} = -\alpha\omega + \beta u + v$$

   where v is white noise of spectral density V.

   (a) Find and plot the Kalman filter gains and corresponding closed-loop poles as a function of the signal-to-noise ratio V/W.

   (b) Using the optimized gains determined in HW #10, problem 2(a) (Friedland 9.6a), find the gain margin for several values of q_1^2 and several values of V/W. Tabulate the results as functions of q_1^2 and V/W

4. (Friedland 11.2) Consider the dynamics for the inverted penulum on a motor-drive cart from HW #9 and HW #10, for which you are to build a full order Kalman filter as an observer.

(a) Assume that the only excitation noise present is coincident with the control and has spectra density v^2, and that the only observation is cart displacement which is measured through white noise of spectra density w^2. Plot the Kalman filter gains and poles as a function of the ratio v^2/w^2 for $1 \leq v^2/w^2 \leq 10^6$ (use at least 6 points in your plot).

(b) Using the gains from HW #10, problem 3(a) (Friedland 9.10a) with $r^2 = 0.01$, determine the compensator transfer functoin $D(s)$ for the range of v^2/w^2 in part (a).

(c) For $r^2 = 0.01$ and $v^2/w^2 = 10^{-3}$ in part (b), determine the range of gains K for which the closed loop system with loop transfer function $KD(s)P(s)$ is stable, where $P(s)$ is the transfer function for the plant.