CALIFORNIA INSTITUTE OF TECHNOLOGY
Control and Dynamical Systems

CDS 110
Problem Set #11

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Due: 29 Jan 03

Note: In the upper left hand corner of the first page of your homework set, please put the number of hours that you spent on this homework set (including reading).

Unless otherwise specified, you may use MATLAB or Mathematica as long as you include a copy of the code used to generate your answer.

1. Consider the optimal control problem for the system

$$\dot{x} = ax + bu$$

   where $x = \mathbb{R}$ is a scalar state, $u \in \mathbb{R}$ is the input, the initial state $x(t_0)$ is given, and $a, b \in \mathbb{R}$ are positive constants. The cost function is given by

$$J = \tfrac{1}{2} \int_{t_0}^{t_f} u^2(t)\, dt + \tfrac{1}{2} c x^2(t_f),$$

   where the terminal time t_f is given and c is a constant.

   Solve explicitly for the optimal control $u^*(t)$ and the corresponding state $x^*(t)$ in terms of t_0, t_f, $x(t_0)$ and t and describe what happens to the terminal state $x^*(t_f)$ as $c \to \infty$. (Hint: use the maximum principle to get the governing equations and solve these explicitly. The only tricky part is figuring out how to deal with the boundary conditions, but remember that everything is a *scalar* and so you can solve things in closed form.)

2. Consider the same system as in the previous problem, but let $t_f = \infty$ and use the cost function

$$J = \tfrac{1}{2} \int_{t_0}^{\infty} u^2(t)\, dt.$$

   Solve for $u^*(t) = -bPx^*(t)$ where P is the positive solution corresponding to the algebraic Riccati equation. Note that this gives an explicit feedback law ($u = -bPx$) while the solution for the previous problem was in terms of the initial condition on x.

   (Note that $Q = 0$ in this problem and hence the observability condition given in class is not satisfied. J has finite minimum in this case because the scalar Riccati equation has a positive solution.)

3. Using the following parameter values

$$a = 2 \qquad b = 0.5 \qquad x(t_0) = 4$$
$$c = 0.1,\, 10 \qquad t_f = 0.5,\, 1,\, 10$$

   implement the controller from problem 1 in a receding horizon fashion with an update time of $\delta = 0.5$. Compare the responses of the receding horizon controllers to the LQR controller designed in problem 2, from the same initial condition. What do you observe as c and t_f increase?

   (Hint: you can write a MATLAB script to do this by performing the following steps:

(i) set $t_0 = 0$

(ii) using the closed form solution for x^* from problem 1, plot $x(t)$, $t \in [t_0, t_f]$ and save $x_\delta = x(t_0 + \delta)$

(iii) set $x(t_0) = x_\delta$ and repeat step (ii) until x is small.)

4. In this problem we will explore the effect of constraints on control of the linear unstable system given by

$$\dot{x}_1 = 0.8x_1 - 0.5x_2 + 0.5u$$
$$\dot{x}_2 = x_1 + 0.5u$$

subject to the constraint that $|u| \leq a$ where a is a postive constant.

(a) Ignoring the constraint ($a = \infty$) and design an LQR controller to stabilize the system. Plot the response of the closed system from the initial condition given by $x = (1, 0)$.

(b) Use SIMULINK or `ode45` to simulate the the system for some finite value of a with an initial condition $x(0) = (1, 0)$. Numerically (trial and error) determine the smallest value of a for which the system goes unstable.

(c) Let $a_{\min}(\rho)$ be the smallest value of a for which the system is stable from $x(0) = (\rho, 0)$. Plot $a_{\min}(\rho)$ for $\rho = 1, 4, 16, 64, 256$.

(d) *Optional*: Given $a > 0$, design and implement a receding horizon control law for this system. Show that this controller has larger region of attraction than the controller designed in part (b). (Hint: solve the finite horizon LQ problem analytically, using the bang-bang example as a guide to handle the input constraint.)