Reference inputs and integral action

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A state space control law of the form

$$u = -K(x - x_d)$$

can be used to stabilize an equilibrium point x_d only if the corresponding reference input, u_d, is zero. This note gives a description of how to include reference inputs and integral action in state space control laws. These two mechanisms can be used to eliminate the steady state error that is usually present with state space feedback.

We consider a linear system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}$ is the input and $y \in \mathbb{R}$ is the output. We let r represent the desired reference value for the output y.

1 Reference inputs

Suppose we have an equilibrium point (x_d, u_d). We augment the normal state space control law to include the nominal input:

$$u = u_d - K(x - x_d).$$

This allows the requisite input $u = u_d$ to be applied when $x = x_d$ and hence the equilibrium point is achieved.

In many cases, we want to automatically adjust the equilibrium point based on the (constant) reference r. To find x_d and u_d as a function of r, we solve the system equations:

$$\begin{aligned} 0 &= Ax_d + Bu_d \\ r &= Cx_d + Du_d. \end{aligned}$$

This is a linear equation, and hence we have

$$\begin{bmatrix} x_d \\ u_d \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} r = \begin{bmatrix} N_x \\ N_u \end{bmatrix} r$$

where $N_x \in \mathbb{R}^n$ and $N_u \in \mathbb{R}$.

We can now rewrite the control law to be of the form

$$u = -K(x - N_x r) + N_u r = -Kx + Nr$$

where

$$N = N_u + KN_x.$$

The scalar N represents a *feedforward* gain. The closed loop system is

$$\begin{aligned}\dot{x} &= (A - BK)x + BNr \\ y &= (C - DK)x + DNr\end{aligned}$$

where r represents the reference input and y is the output for the closed loop plant. If N is chosen correctly, the steady state error for the closed loop system should be zero.

It can be shown (homework exercise) that this approach is not very robust and that, in particular, if the system model is not correct then this approach will generate steady state errors. One way of seeing this is that the resulting control law does not compare the actual output to the reference output, and hence it cannot correct for errors in the output.

2 Integral action

An alternative approach to achieving zero steady state error is to add integral action to a state space feedback controller. We start by augmenting the plant dynamics with an extra state x_i that is the integral of the output error:

$$\begin{aligned}\dot{x} &= Ax + Bu \\ \dot{x}_i &= Cx - r \\ y &= Cx.\end{aligned}$$

This can be written in terms of the augmented state (x, x_i):

$$\frac{d}{dt}\begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r.$$

We design a control law for the case where $r = 0$ to get a state feedback

$$u = -\begin{bmatrix} K & K_i \end{bmatrix} \begin{bmatrix} x - x_d \\ x_i \end{bmatrix} + u_d$$

that places the closed loop poles at the desired location (using either pole place or LQR). This technique has the advantage of being robust to variations in the system matrix (A, B, C, D) that are used to compute the equilibrium states x_d and u_d. In particular, the controller will automatically adjust the integrator state to provide sufficient input to hold the output at the reference value.