Announcements:

1. Final available, due W 3/19 by 5pm

2. HW #15 graded by Thursday pm, soln's posted

3. HW #16 graded by Mon., soln's posted this Friday.

4. Please fill out surveys & return to ______

CDS 110b Final Review Notes

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1 General Comments

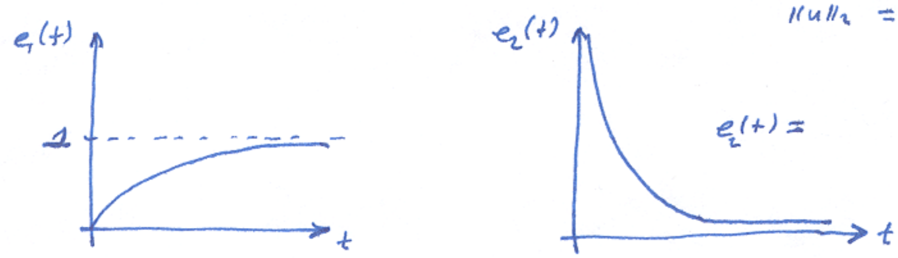
The exam will cover mostly material since the midterm, but may include or rely on concepts from the first half of the term. As usual, the best resources for review are

- Past Homeworks... #'s 12-16
- Lecture Notes...
- Text Reading and Examples...

2 DFT Chapter 2: Norms for Signals and Systems

In this chapter, the temporal norms of time-varying signals are presented as a way to measure their size. Also, the notion of a norm for a system's transfer function is introduced. Consider two signals $e_1(t)$ and $e_2(t)$:

$\|u\|_2 = \left(\int_{-\infty}^{\infty} |u(t)|^2 dt \right)^{\frac{1}{2}}$



$\|e_1\|_\infty = 1$ $\qquad$ $\|e_2\|_\infty = \infty$

$\|e_1\|_2 = \infty$ $\qquad$ $\|e_2\|_2 =$ (const.)

For $e_1(t)$ the infinity-norm (peak) is 1, $\|e_1(t)\|_\infty = 1$, whereas since the signal does not "die out" the 2-norm is infinite, $\|e_1(t)\|_2 = \infty$. For $e_2(t)$ the opposite is true.

Consider a system G with input d and output e,

$$e = Gd$$

For performance we may want the output signal e to be small for any allowed input signals d, i.e. we need to specify the set that d belongs to and what we mean by "small". Possible input signals

*These notes adapted from "CDS 212 Revision notes", parts 1 and 2 by Antonis Papachristodoulou. Discrete time system notes contributed by Zhipu Jin. DFT refers to Doyle, Francis and Tannenbaum. *Feedback Control Theory.* Macmillan. 1992.

sets might be specific (may be white noise process with zero mean, a sinusoid of fixed frequency etc) or they might be specified by norm bounds (bounded in energy (2-norm) in power (power "norm"), in magnitude (∞-norm)). To measure the output signal one may consider the 1-norm, the 2-norm (energy), the ∞-norm or the power.

The question to ask is what is the appropriate system gain to test for performance. You should be familiar with induced norms and translating signal specifications into equivalent specifications on system norms.

Example 2.1. Let $u(t)$ be a continuous signal whose derivative $\dot{u}(t)$ is also continuous. Which of the following qualify as a norm for u?

i) $\|u\| \geq 0$
ii) $\|u\| = 0 \Leftrightarrow u(t) \equiv 0$
iii) $\|au\| = |a|\,\|u\|$ $\quad a \in \mathbb{R}$
iv) $\|u+v\| = \|u\| + \|v\|$

- $\sup_t |\dot{u}(t)|$ $\qquad u(t) = 1 \Rightarrow \|u\| = 0$ violates (ii)
- $|u(0)| + \sup_t |\dot{u}(t)|$

(i) $\|u\| \geq 0$ ✓

(ii) $u(t) = 0 \Rightarrow \|u\| = 0$ ✓

$\|u\| = 0 \Rightarrow u(0) = 0$ & $\dot{u}(t) = 0 \ \forall t \Rightarrow u(t) = const = u(0)$ ✓

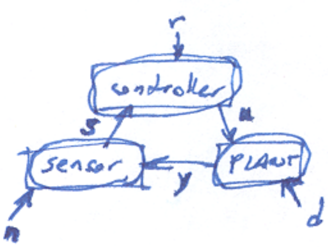
(iii) $\|au\| = |a\,u(0)| + \sup_t |a\,\dot{u}(t)| = a\,\|u\|$ ✓

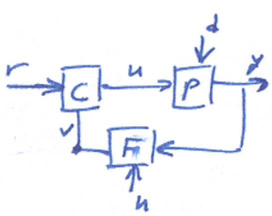
(iv) $\|u+v\| = |u(0) + v(0)| + \sup_t |\dot{u}(t) + \dot{v}(t)| \leq$

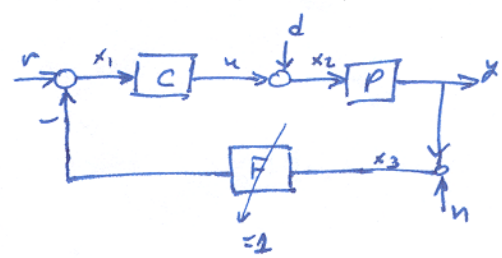
$|u(0)| + |v(0)| + \sup_t |\dot{u}(t)| + \sup_t |\dot{v}(t)| = \|u\| + \|v\|$

3 DFT Chapter 3: Basic Concepts

Standard Block Diagrams For a plant P, controller C, disturbance d, noise n, ...







Theorem 3.1 (Nyquist Stability Criterion). *Let N be the number of poles of $G(s)$ in the RHP. Then $\frac{1}{1+\alpha G}$ is stable iff the number of anticlockwise encirclements of the $-1/\alpha$ point equals N.*

Internal Stability Consider the plant

$$P(s) = P_1(s)\frac{2-s}{2+s}$$

and a proposed controller

$$C(s) = C_1(s)\frac{2+s}{2-s}$$

where C_1 is a suitable compensator for P_1. Now $PC = P_1C_1$, so if $\frac{1}{1+P_1C_1}$ is stable, $\frac{1}{1+PC}$ will be too. However we must ensure that the response of the control input to disturbances is also stable, and in this case we find that

$$\frac{C}{1+CP} = \frac{C_1}{1+C_1P_1}\frac{2+s}{2-s}$$

which is unstable. In general, a feedback system comprising a plant P and a controller C is internally stable if each of the following closed-loop transfer functions is stable:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{1}{1+PC\cancel{F}} \begin{pmatrix} 1 & -P\cancel{F} & -\cancel{F} \\ C & 1 & -C\cancel{F} \\ PC & P & 1 \end{pmatrix} \begin{pmatrix} r \\ d \\ n \end{pmatrix}$$

distinct

In fact, we only need to check if two of them are. Which two?

$$\frac{C}{1+PC}, \quad \frac{P}{1+PC}$$

This condition is equivalent to any one of them being stable (i.e. $1 + PC$ having no zeros in the RHP) and there being no closed RHP zero/pole cancellations between P and C.

4 DFT Chapter 4: Uncertainty and Robustness

Recall that making the sensitivity function small is one of the most fundamental objectives of feedback design. Consider $T(s)$ as a function of P. Then the closed loop transfer function (complementary sensitivity)

$$T = \frac{PC}{1+PC}$$

has some perturbation due to infinitesimal changes in the plant which is

$$\frac{lo\, d\, hi - hi\, d\, lo}{lo^2}$$

$$\lim_{\Delta P \to 0} \frac{\Delta T/T}{\Delta P/P} = \frac{dT}{dP}\frac{P}{T} = \frac{(1+\cancel{PC})C - \cancel{PC}(C)}{(1+PC)^{\cancel{2}}} \cdot \frac{\cancel{P}\,\cancel{(1+PC)}}{\cancel{PC}} = \frac{1}{1+PC}$$

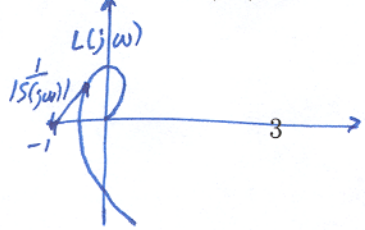
$$= S \quad \text{(sensitivity function)}$$

This means that the relative error in the closed-loop transfer function is reduced compared to the relative error in the open loop transfer function if $\|S(s)\|_\infty < 1$, or when we introduce a weight, $\|W_1S\|_\infty < 1$. This requires that at every frequency the point $L(j\omega)$ on the Nyquist plot lies outside the disk of center -1, radius $W_1(j\omega)$.

$|S| = \left|\frac{1}{1+L}\right|$

$\Leftrightarrow \frac{1}{|S|} = |1+L|$

$\Rightarrow$

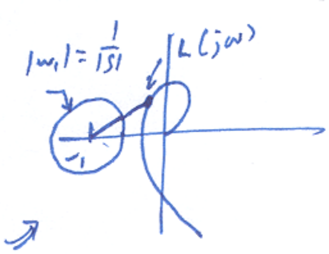


$|W_1 S(j\omega)| < 1$

$\Leftrightarrow |S(j\omega)| < \frac{1}{|W_1|}$

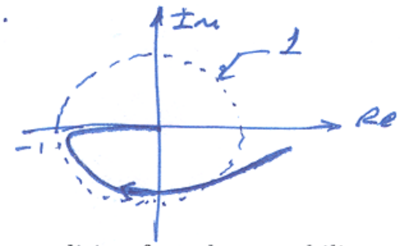
$\Leftrightarrow \frac{1}{|S(j\omega)|} > |W_1|$

$\Leftrightarrow |1+L| > |W_1| \Rightarrow$



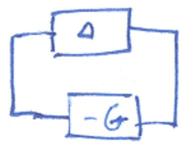
Example 4.1. We all know the notion of phase and gain margin. True or false: $\|S(s)\|_\infty$ will become large if and only if our system has poor gain margin or poor phase margin.

COUNTEREXAMPLE.



We will use the Small Gain Theorem to find a condition for robust stability:

Theorem 4.1. *Suppose $G(s)$ is a stable fixed transfer function. Let γ be a positive constant. Then the feedback system*



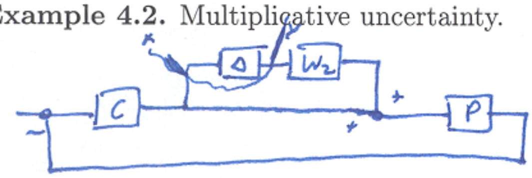
is stable for all stable $\Delta(s)$ satisfying $|\Delta(j\omega)| \leq \gamma$ iff $\|G(j\omega)\|_\infty < 1/\gamma$.

Consider now a multiplicative uncertainty model, assuming the feedback system is stable. Write

$$\tilde{P} = (1 + W_2\Delta)\, P$$

where W_1, P are fixed, Δ is not, but stable and norm bounded: $\|\Delta\|_\infty \leq 1$.

Example 4.2. Multiplicative uncertainty.



$x = -PC(x + W_2 y)$

$\Rightarrow x(1 + PC) = -W_2 PC y$

$\Rightarrow \frac{x}{y} = -W_2 T = -G$

$\Rightarrow G = W_2 T$

$\Rightarrow$ ROBUST STABILITY $\|W_2 T\|_\infty < 1$

NOM PERFORMANCE: $\|W_1 S\|_\infty < 1$

RS: $\|W_2 T\|_\infty < 1$

RP $\Leftrightarrow$ $\| |W_1 S| + |W_2 T| \|_\infty < 1$ (PROOF IN DFT SEC 4.6)

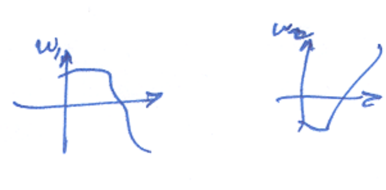
5 DFT Chapter 6: Design Constraints

Recall the following constraints:

- Algebraic Constraints:
  1. $\mathbf{S} + \mathbf{T} = \mathbf{1}$: At any frequency ω, $|S(j\omega)|$ and $|T(j\omega)|$ cannot be both less than 0.5.
  2. The weighting functions $W_1(j\omega)$ and $W_2(j\omega)$ satisfy

$\min \{ |W_1(j\omega)|, |W_2(j\omega)| \} < 1 \quad \forall \omega$

Typ:



Recall that W_1 is monotonically decreasing for good tracking of low-frequency signals, and W_2 is monotonically increasing, as uncertainty increases at high frequencies.

3. If p is a pole and z is a zero of L in the RHP, then

$$S(p) = 0 \qquad S(z) = 1 \qquad S = \frac{1}{1+PC}$$

$$T(p) = 1 \qquad T(z) = 0 \qquad T = \frac{PC}{1+PC}$$

- Analytic Constraints:

1. If L has a zero z in RHP then ← nonminimum phase

$$\|W_1 S\|_\infty \geq |W_1(z)|$$

2. If L has a pole p in RHP then

$$\|W_2 T\|_\infty \geq |W_2(p)|$$

} Maximum modulus principle

3. If L has a pole p in RHP and a zero z in RHP then

Consider case where only one zero & one pole in RHP

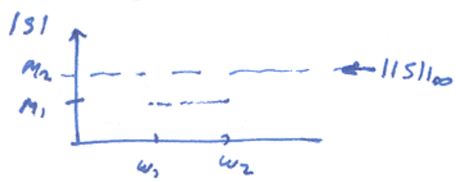
① $S = S_{ap} S_{mp}$ ← no RHP zeros, $|S_{ap}| = 1 \ \forall \omega$, $T = T_{ap} T_{mp}$

② $S_{ap} = \frac{s-p}{s+p}$, $T_{ap} = \frac{s-z}{s+z}$

③ $\|W_1 S\|_\infty = \|W_1 S_{mp}\|_\infty = \|W_1 S_{ap}^{-1}\|_\infty \geq |W_1(z) S_{ap}^{-1}(z)| = |W_1(z)| \left|\frac{z+p}{z-p}\right|$

$\|W_2 T\|_\infty = \|W_2 T_{mp}\|_\infty = \|W_2 T_{ap}^{-1}\|_\infty \geq |W_2(p)| \left|\frac{p+z}{p-z}\right|$

4. The Waterbed effect: Let $M_1 := \max_{\omega_1 \leq \omega \leq \omega_2} |S(j\omega)|$ and M_2 the maximum magnitude over all frequencies, i.e. $\|S\|_\infty$. If L has a zero z in the RHP then there exist c_1 and c_2 depending on ω_1, ω_2 and z such that



$$c_1 \log M_1 + c_2 \log M_2 \geq \log |S_{ap}(z)^{-1}| \geq 0$$

uses Poisson integral formula; proof in section 6.2 of DFT

5. The area formula: Let p_i denote the set of poles of L in RHP.

$$\int_0^\infty \log |S(j\omega)| \, d\omega = \pi \log e \left(\sum Re\{p_i\} \right)$$

Exercise 5.1. Let ω be a frequency such that $j\omega$ is not a pole of P. Suppose that $\epsilon := |S(j\omega)| < 1$ at this frequency. Derive a lower bound for $|C(j\omega)|$ that blows up as $\epsilon \to 0$.

$$\epsilon = \frac{1}{|1+PC|} \Rightarrow \frac{1}{\epsilon} = |1+PC| \leq 1 + |P||C|$$

$$\Rightarrow |C| \geq \frac{1}{|P|} \frac{(1-\epsilon)}{\epsilon}$$

6 DFT Chapter 7: Loopshaping

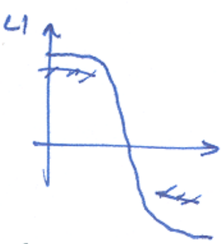
The idea is to construct a **stable** loop transfer function L such that

$$\| |W_1 S| + |W_2 T| \|_\infty < 1$$

is satisfied and then get $C = L/P$. To do this, require . . . (typically)

for low frequency $|W_1| >> 1 > |W_2|$, $\quad |L| > \frac{|W_1|}{1-|W_2|}$

for high frequency $|W_1| < 1 << |W_2|$, $\quad |L| < \frac{1-|W_1|}{|W_2|}$

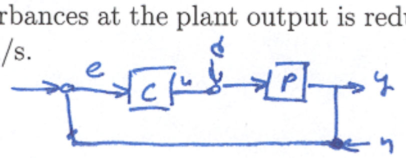


and shape L accordingly, until you get an L that satisfies robust stability (make sure it's feedback is stable). In general, if L rolls off with more than 40 dB/dec at the crossover frequency the closed loop system is likely to be unstable.

7 Some Brief Exercises

Translate the following specifications for the closed-loop system.

1. The effect of the plant input disturbances at the plant output is reduced to ε times their open loop magnitude for $0 \leq \omega \leq 1$ rad/s.



$y = P(d - Cy)$

$\Rightarrow H_{yd} = \frac{P}{1+PC}$

2. Sensor noise should be reduced at the plant output by at least 40dB for $\omega \geq 10$ rad/s

$y = -PC(n+y)$ $\quad |T| < 0.01 \quad \omega \geq 10 \frac{rad}{s}$

$H_{yn} = -\frac{PC}{1+PC} = -T$

$|S+T| = 1$

$|S| + |T| \geq |S+T| = 1$

$\Rightarrow |S| \geq 1 - |T|$

Suppose that $|S(j\omega)| \leq 1.46$ for all ω. Show that this is sufficient to ensure a phase margin of at least 40°.

~~$\Rightarrow S = T - 1$~~

~~$|S| \leq |T| + 1$~~

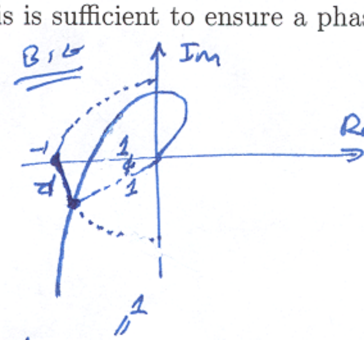
~~In this case~~

~~$|S| \geq 0.99$~~

~~$|S| \leq 1.01$~~

$$\frac{1}{|S(j\omega)|} = |1+L|$$

$$\frac{1}{\|S(j\omega)\|_\infty} = \inf_\omega |1 + L(j\omega)| \geq \frac{1}{1.46}$$



$d \geq R \sin \varphi$

$\Rightarrow \varphi \geq \text{Arcsin}\left(\frac{1}{1.46}\right) = 0.7545$

$= 43.23$

8 Discrete Time Systems

z-Transform and Difference Equations

1. Definition $Z\{f(k)\} = \sum_{k=0}^{\infty} f(k) z^{-k}$

2. Get z transfer function from difference equations

$$Z\{f(k-1)\} = z^{-1}F(z),\ if\ Z\{f(k)\} = F(z)$$

   For difference equations

$$y(k) = -a_1 y(k-1) - a_2 y(k-2) + b_0 u(k) + b_1 u(k-1),$$

   the z transfer function is

$$H(z) = \frac{Y(z)}{U(z)} = \frac{b_0 + b_1 z^{-1}}{1 + a_1 z^{-1} + a_2 z^{-2}}.$$

   For state space system

$$\begin{cases} x(k+1) &= A_d x(k) + B_d u(k) \\ y(k) &= C_d x(k) + D_d u(k) \end{cases},$$

   the z transfer function is $H(z) = C_d(z \cdot I - A_d)^{-1} B_d + D_d$.

3. Final value theorem of z transfer function:

$$\lim_{k\to\infty} h(k) = \lim_{z\to 1} (1 - z^{-1}) H(z).$$

4. The stable area of discrete systems is $|z| < 1$ in the z-plane. You can use the same skills as the continuous systems to design compensator, relocate poles, set up observers, etc.

Continuous to Discrete Transformation

1. s-transfer function to z-transfer function by substitute every s as $\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}$. This can be done by MATLAB command as `c2d(tf, T, 'tustin')` which is based on bilinear approximation (Tustin's method).

2. State space case. There are three methods mentioned in class. Assume a continuous system is

$$\dot{x} = A_c x + B_c u$$
$$y = C_c x + D_c u$$

   (a) Forward Difference

$$x(k+1) = (I + TA_c)x(k) + TB_c u(k)$$
$$y(k) = C_c x(k)$$

   Note: c2d(sys, T, 'ZOH') is not totally equal to forward difference method.

(b) Backward Difference

$$x(k+1) = (I - TA_c)^{-1}x(k) + (I - TA_c)^{-1}TB_c u(k)$$
$$y(k) = C_c x(k)$$

(c) Bilinear Approximation (Tustin's Method)

$$x(k+1) = (I - 0.5TA_c)^{-1}(I + 0.5TA_c)x(k) + 0.5T(I - 0.5TA_c)^{-1}(I + 0.5TA_c)B_c u(k)$$
$$y(k) = C_c x(k)$$

9 Additional Notes