1. Consider the normalized dynamics of an inverted pendulum described by the model

\[ \frac{dx_1}{dt} = x_2, \quad \frac{dx_2}{dt} = \sin x_1 + u \cos x_1, \]

where \( x_1 \) is the angular deviation from the upright position (\( \theta \)), \( x_2 \) is the angular rate (\( \dot{\theta} \)) and \( u \) is the (scaled) acceleration of the pivot, as shown in Figure 5.16a.

(a) Show that the linearization of the dynamics around the (upward-pointing) equilibrium point \( x_e = (0, 0) \) is given by

\[ \begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u = Ax + Bu. \]

(b) Compute the eigenvalues of the linearized dynamics and show that this equilibrium point is unstable.

(c) Compute the reachability matrix for the system and show that the linearized system is reachable.

(d) Determine a state feedback control law of the form \( u = -Kx \) that gives a closed loop system with the characteristic polynomial \( s^2 + 2\zeta_0\omega_0 s + \omega_0^2 \).

(e) Set \( \omega_0 = 1 \) and \( \zeta_0 = 0.5 \). Compute the eigenvalues for the resulting closed loop system and verify that the equilibrium point is no longer unstable.

(f) Simulate the response of the original nonlinear system from a set of initial conditions that each correspond to the system starting at rest (\( \dot{\theta} = 0 \)) with initial angle \( \theta(0) \) equal to 0.1 rad, 0.5 rad, 1 rad and 2 rad.
1. Consider the normalized, linearized inverted pendulum which is described by
\[
\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u = Ax + Bu, \quad y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = Cx
\]

Determine a state feedback and reference gain \( u = -Kx + kr \) that gives a closed loop system with unit static gain (steady-state output \( y = r \)) and with the characteristic polynomial \( s^2 + 2\zeta\omega_0 s + \omega_0^2 \).

2. Consider the system
\[
\frac{dx}{dt} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u, \quad y = \begin{bmatrix} 1 & 0 \end{bmatrix} x,
\]

with the control law \( u = -k_1 x_1 - k_2 x_2 + kr \).

Compute the rank of the reachability matrix for the system and show that eigenvalues of the system cannot be assigned to arbitrary values.

Note: You should explicitly show that not all eigenvalues can be assigned arbitrarily; don’t just cite Theorem 6.1.)

3. (Åström and Murray, Exercise 7.10) Let \( A \in \mathbb{R}^{n \times n} \) be a matrix with characteristic polynomial \( \lambda(s) = \det(sI - A) = s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n \). Assume that the matrix \( A \) can be diagonalized and show that it satisfies
\[
\lambda(A) = A^n + a_1 A^{n-1} + \cdots + a_{n-1} A + a_n I = 0,
\]

Use the result to show that \( A^k, k \geq n \), can be rewritten in terms of powers of \( A \) of order less than \( n \).

Note: Assume that the \( A \) matrix is diagonalizable (the theorem is valid but hard to prove with a non-trivial Jordan form).

4. (Åström and Murray, Exercise 7.13) Consider the Whipple bicycle model given by equation (4.7) in Section 4.2 Using the parameters from the companion web site, the model
is unstable at the velocity $v = 5 \text{ m/s}$ and the open loop eigenvalues are $-1.84$, $-14.29$ and $1.30 \pm 4.60i$. Find the gains of a controller that stabilizes the bicycle and gives closed loop eigenvalues at $-2$, $-10$ and $-1 \pm i$. Simulate the response of the system to a step change in the steering reference of 0.002 rad.

Next, find the controller gains corresponding to choosing the final pair of complex poles at $-2 \pm 2i$ and $-5 \pm 5i$. In addition to calculating the state feedback gains, make sure to solve for the reference gain $k_r$ as well. For each case, simulate the response to a step change in the steering reference of 0.002 rad and plot both the steering angle and the torque command.

Note: Download the file `bike_linmod.m` from the course web site, which contains the parameters for the bicycle and generates the matrices $M$, $C$, $K_0$ and $K_2$ in equation (4.7) of the text.

Find the controller gains corresponding to choosing the final pair of complex poles at $-1 \pm i$ as stated in the text, and also with these poles at $-2 \pm 2i$ and $-5 \pm 5i$. In addition to calculating the state feedback gains, solve for the reference gain $k_r$ as well! When simulating the response to a step change in the steering reference of 0.002 rad, plot both the steering angle output $\delta$ and the torque command.