CALIFORNIA INSTITUTE OF TECHNOLOGY
Control and Dynamical Systems

CDS 101

R. Murray | **Problem Set #3** | Issued: 12 Oct 2015 (v2)
Fall 2015 | | Due: 21 Oct 2015, 2 pm

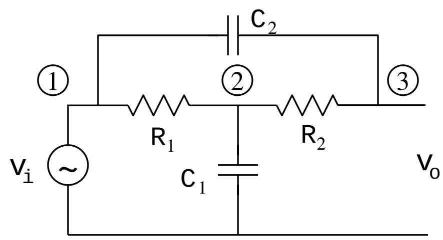
Note: In the upper left hand corner of the *second* page of your homework set, please put the number of hours that you spent on this homework set (including reading).

Reminder: You get two grace periods of no more than 2 days each for late homework. After that, late homework will not be accepted without a note from the dean or the health center.

1. For one of the following linear systems, determine whether the origin is asymptotically stable and, if so, plot the step response and frequency response for the system. If there are multiple inputs or outputs, plot the response for each pair of inputs and outputs.

   (a) *Coupled mass spring system.* Consider the coupled mass spring system from Example 6.6 with $m = 250$, $k = 50$ and $c = 10$. The input $u(t)$ is the force applied to the right-most spring and the outputs are the positions q_1 and q_2.

   (b) *Bridged Tee Circuit.* Consider the following electrical circuit, with input v_i and output $y = v_o$.



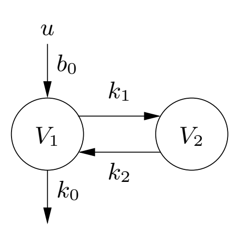
   The dynamics are given by

$$\frac{d}{dt}\begin{pmatrix} v_{c1} \\ v_{c2} \end{pmatrix} = \begin{pmatrix} -\frac{1}{C_1}\left(\frac{1}{R_1}+\frac{1}{R_2}\right) & -\frac{1}{C_1 R_2} \\ -\frac{1}{C_2 R_2} & -\frac{1}{C_2 R_2} \end{pmatrix}\begin{pmatrix} v_{c1} \\ v_{c2} \end{pmatrix} + \begin{pmatrix} \frac{1}{C_1}\left(\frac{1}{R_1}+\frac{1}{R_2}\right) \\ \frac{1}{C_2 R_2} \end{pmatrix} v_i,$$

$$y = \begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} v_{c1} \\ v_{c2} \end{pmatrix} + v_i,$$

   where v_{c1} and v_{c2} are the voltages across the two capacitors. Assume that $R_1 = 100\ \Omega$, $R_2 = 100\ \Omega$ and $C_1 = C_2 = 1 \times 10^{-6}$ F.

   (c) *Compartment model.* Consider the two-compartment model described in Section 4.6 and shown below.



The dynamics for this system can be written as

$$\frac{dc}{dt} = \begin{pmatrix} -k_0 - k_1 & k_1 \\ k_2 & -k_2 \end{pmatrix} c + \begin{pmatrix} b_0 \\ 0 \end{pmatrix} u, \qquad y = \begin{pmatrix} 0 & 1 \end{pmatrix} c.$$

Use the parameter values $k_0 = 0.1$, $k_1 = 0.1$, $k_2 = 0.5$ and $b_0 = 1.5$.

2. Consider the balance system described in Example 2.1 of the text, using the following parameters:

$$\begin{aligned} M &= 10 \text{ kg}, & m &= 80 \text{ kg}, & J &= 100 \text{ kg m}^2, \\ c &= 0.1 \text{ N/m/sec}, & l &= 1 \text{ m}, & \gamma &= 0.01 \text{ Nms}, \end{aligned} \qquad g = 9.8 \text{ m/s}^2.$$

A nonlinear simulation of this system is available on the course webpage, using ODE45. (This system has also been modeled in SIMULINK in the file `balance_simple.mdl`, available from the course web page; if you wish, you are welcome to use that in answering part (b) below).

For the system linearization, there are some small glitches in the equations listed in the text (noted in the errata) in both Example 2.1 ($A(3,4)$ should not have J_t) and in Example 6.7 ($A(4,4)$ involves M_t, not J_t).) The correct linearization is

$$\frac{d}{dt}\begin{pmatrix} p \\ \theta \\ \dot{p} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & m^2\ell^2 g/\mu & -cJ_t/\mu & -\gamma\ell m/\mu \\ 0 & M_t mg\ell/\mu & -c\ell m/\mu & -\gamma M_t/\mu \end{pmatrix} \begin{pmatrix} p \\ \theta \\ \dot{p} \\ \dot{\theta} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ J_t/\mu \\ m\ell/\mu \end{pmatrix} u$$

(a) We can design a stabilizing control law for this system using "state feedback", which is a control law of the form $u = -Kx$ (we will learn about this more next week). The closed loop system under state feedback has the form

$$\frac{dz}{dt} = (A - BK)z.$$

Show that the following state feedback stabilizes the linearization of the inverted pendulum on a cart: $K = [-15.3\ 1730\ \ -50\ 443]$.

(b) Now build a simulation for the closed loop, *nonlinear* system (either using ODE45 or SIMULINK; in either case, you should look in the relevant file and try to understand how it works). Simulate several different initial conditions and show that the controller *locally* asymptotically stabilizes the system to x_e from these initial conditions. Include plots of a representative simulation for an initial condition that is in the region of attraction of the controller and one that is outside the region of attraction.

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CDS 110a

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1. For two of the following linear systems, determine whether the origin is asymptotically stable and, if so, plot the step response and frequency response for the system. If there are multiple inputs or outputs, plot the response for each pair of inputs and outputs.

   (a) *Coupled mass spring system.* Consider the coupled mass spring system from Example 6.6 with $m = 250$, $k = 50$ and $c = 10$. The input $u(t)$ is the force applied to the right-most spring and the outputs are the positions q_1 and q_2.

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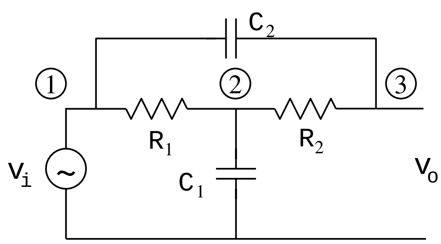

   The dynamics are given by

$$\frac{d}{dt}\begin{pmatrix} v_{c1} \\ v_{c2} \end{pmatrix} = \begin{pmatrix} -\frac{1}{C_1}\left(\frac{1}{R_1}+\frac{1}{R_2}\right) & -\frac{1}{C_1 R_2} \\ -\frac{1}{C_2 R_2} & -\frac{1}{C_2 R_2} \end{pmatrix}\begin{pmatrix} v_{c1} \\ v_{c2} \end{pmatrix} + \begin{pmatrix} \frac{1}{C_1}\left(\frac{1}{R_1}+\frac{1}{R_2}\right) \\ \frac{1}{C_2 R_2} \end{pmatrix} v_i,$$

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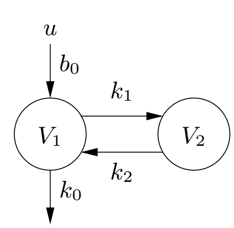

The dynamics for this system can be written as

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(a) We can design a stabilizing control law for this system using "state feedback", which is a control law of the form $u = -Kx$ (we will learn about this more next week). The closed loop system under state feedback has the form

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3. Consider a first-order system of the form

$$\tau \frac{dx}{dt} = -x + u, \qquad y = x.$$

We say that the parameter τ is the *time constant* for the system since the zero input system approaches the origin as $e^{-t/\tau}$. For a first-order system of this form, show that the rise time for a step response of the system is approximately 2τ, and that 1%, 2%, and 5% settling times approximately corresponds to 4.6τ, 4τ and 3τ.

4. Consider a linear discrete-time system of the form

$$x[k+1] = Ax[k] + Bu[k], \qquad y[k] = Cx[k] + Du[k].$$

(a) Show that the general form of the output of a discrete-time linear system is given by the discrete-time convolution equation:

$$y[k] = CA^k x[0] + \sum_{j=0}^{k-1} CA^{k-j-1} Bu[j] + Du[k].$$

(b) Show that a discrete-time linear system is asymptotically stable if and only if all the eigenvalues of A have a magnitude strictly less than 1.

(c) Let $u[k] = \sin(\omega k)$ represent an oscillatory input with frequency $\omega < \pi$ (to avoid "aliasing"). Show that the steady-state component of the response has gain M and phase θ, where

$$Me^{i\theta} = C(e^{i\omega} I - A)^{-1} B + D.$$

Hint: For part (a), note that the equation can also be written as

$$y[k] = CA^k x_0 + \sum_{i=0}^{k-1} CA^i Bu[k-1-i] + Du[k]$$

For part (b), you can assume that the matrix A has a full basis of eigenvectors. For part (c), it is sufficient to compute the response to $u[k] = e^{i\omega k}$ and skip the step of then calculating the response to $\sin \omega k = \frac{1}{2i}(e^{i\omega k} - e^{-i\omega k})$. You should also assume that the system is asymptotically stable. Finally, you may also want to use the Taylor series expansion:

$$(I - X)^{-1} = I + X + X^2 + \cdots$$