CALIFORNIA INSTITUTE OF TECHNOLOGY
Computing and Mathematical Sciences

CDS 101

R. Murray — **Problem Set #2** — Issued: 5 Oct 2015 (v2)
Fall 2015 — Due: 14 Oct 2015, 2 pm

Note: In the upper left hand corner of the *second* page of your homework set, please put the number of hours that you spent on this homework set (including reading).

Reminder: You get two grace periods of no more than 2 days each for late homework. After that, late homework will not be accepted without a note from the dean or the health center.

1. Choose any *one* of the following systems, locate the equilibrium points for the system and indicate whether each is asymptotically stable, stable (but not asymptotically stable) or unstable. To determine stability, you can either use a phase portrait (if appropriate), analyze the linearization or simulate the system using multiple nearby initial conditions to determine how the state evolves.

   (a) *Nonlinear spring mass.* Consider a nonlinear spring mass system with dynamics

   $$m\ddot{q} = -k(q - aq^3) - c\dot{q},$$

   where $m = 1000$ kg is the mass, $k = 250\ \mathrm{kg/s^2}$ is the nominal spring constant, $a = 0.01$ represents the nonlinear "softening" coefficient of the spring and $c = 100$ kg/s is the damping coefficient. Note that this is very similar to the spring mass system we have studied in Section 3.2, except for the nonlinearity.

   (b) *Genetic toggle switch.* Consider the dynamics of two repressors connected together in a cycle. It can be shown (Exercise 3.9) that the normalized dynamics of the system can be written as

   $$\frac{dz_1}{d\tau} = \frac{\mu}{1 + z_2^n} - z_1 - v_1, \qquad \frac{dz_2}{d\tau} = \frac{\mu}{1 + z_1^n} - z_2 - v_2.$$

   where z_1 and z_2 represent scaled versions of the protein concentrations, v_1 and v_2 represent external inputs and the time scale has been changed. Let $\mu = 2.16$, $n = 2$ and $v_1 = v_2 = 0$.

   (c) *Congestion control of the Internet.* A simplified model for congestion control between N computers connected by a router is given by the differential equation

   $$\frac{dx_i}{dt} = -b\frac{x_i^2}{2} + (b_{\max} - b), \qquad \frac{db}{dt} = \left(\sum_{i=1}^{N} x_i\right) - c,$$

   where $x_i \in \mathbb{R}$, $i = 1, \dots, N$ are the transmission rates for the sources of data, $b \in \mathbb{R}$ is the current buffer size of the router, $b_{\max} > 0$ is the maximum buffer size and $c > 0$ is the capacity of the link connecting the router to the computers. The $\dot{x}_i$ equation represents the control law that the individual computers use to determine how fast to send data across the network and the $\dot{b}$ equation represents the rate at which the buffer on the router fills up. Consider the case where $N = 2$ (so that we have three states, x_1, x_2 and b) and take $b_{\max} = 1$ Mb and $c = 2$ Mb/s.

2. Consider the cruise control system described in Section 4.1. Generate a phase portrait for the closed loop system on flat ground ($\theta = 0$), in third gear, using a PI controller (with $k_p = 0.5$ and $k_i = 0.1$), $m = 1000$ kg and desired speed 20 m/s. Your system model should include the effects of saturating the input between 0 and 1. (Hint: Keep in mind that when modeling feedback control, additional states can arise that do not appear in the original dynamics. You should include the MATLAB code used to generate your phase portrait.)

   Hint: Pay attention to the range over which you plot the phase portrait, e.g., from 15 to 25 m/s captures the "interesting" part of the velocity state.

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1. Consider the linear ordinary differential equation (3.7). Show that by choosing a state space representation with $x_1 = y$, the dynamics can be written as

$$A = \begin{pmatrix} 0 & 1 & & 0 \\ 0 & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \\ -a_n & -a_{n-1} & & -a_1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & \ldots & 0 & 0 \end{pmatrix}.$$

This canonical form is called the *chain of integrators* form.

2. Choose any *two* of the following systems, locate the equilibrium points for the system and indicate whether each is asymptotically stable, stable (but not asymptotically stable) or unstable. To determine stability, you can either use a phase portrait (if appropriate), analyze the linearization or simulate the system using multiple nearby initial conditions to determine how the state evolves.

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   where z_1 and z_2 represent scaled versions of the protein concentrations, v_1 and v_2 represent external inputs and the time scale has been changed. Let $\mu = 2.16$, $n = 2$ and $v_1 = v_2 = 0$.

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   Hint: Pay attention to the range over which you plot the phase portrait, e.g., from 15 to 25 m/s captures the "interesting" part of the velocity state.

4. Consider the linear system

$$\frac{dx}{dt} = \begin{pmatrix} 0 & 1 \\ 0 & -3 \end{pmatrix} x + \begin{pmatrix} -1 \\ 4 \end{pmatrix} u, \qquad y = \begin{pmatrix} 1 & 0 \end{pmatrix} x,$$

   with the feedback $u = -ky$. Plot the location of the eigenvalues as a function the parameter k. Identify the approximate gains at which the system becomes unstable and label these on your plot. (To create your plot, you should compute the eigenvalues at multiple values of k and plot these on the complex plane. Label the locations of the eigenvalues for $k = 0$ with '$\times$' and the locations for $k \to \infty$ with an '$\circ$' if they converge to a finite value. Choose the units on your graph so that key features are visible and use arrows on your plot to indicate which direction corresponds to increasing gain, similar to Figure 5.18b in the text.)