Design Patterns for Control Systems

“Classical” control (1950s...)  
- Reference input shaping  
- Feedback on output error  
- Compensator dynamics shape closed loop response  
- Uncertainty in process dynamics $P(s) +$ external disturbances ($d$) & noise ($n$)

- Goal: output $y(t)$ should track reference trajectory $r(t)$  
- Design typically done in “frequency domain” (second half of CDS 101/110)

“Modern” (state space) control (1970s...)  
- Assume dynamics are given by linear system, with known $A, B, C, D$ matrices  
- Measure the state of the system and use this to modify the input  
- $u = -K \dot{x} + k_r r$

- Goal unchanged: output $y(t)$ should track reference trajectory $r(t)$ [often constant]
Input/Output Control Design Specifications

Keep track all input/output transfer functions
- Keep error small for all reference signals \( r \)
- Attenuate effect of sensor noise \( n \) and disturbances \( d \)
- Avoid large input commands \( u \)

Design represents a tradeoff between the quantities
- Keep \( L = PC \) large for good performance \( (H_{er} << 1) \)
- Keep \( L = PC \) small for good noise rejection \( (H_{ge} < 1) \)

\[ F(s) = 1: \text{Four unique transfer functions define performance ("Gang of Four")}\]
- Stability is always determined by \( 1/(1+PC) \) assuming stable process & controller
- Numerator determined by forward path between input and output

More generally: 6 primary transfer functions; simultaneous design of each
- Controller \( C(s) \) enters in multiple places \( \Rightarrow \) hard to understand tradeoffs

\[
\begin{bmatrix}
\eta \\
y \\
u
\end{bmatrix} =
\begin{bmatrix}
\frac{P}{1+PC} & \frac{PC}{1+PC} & \frac{PCF}{1+PC} \\
\frac{P}{1+PC} & \frac{1}{1+PC} & \frac{PCF}{1+PC} \\
-\frac{PC}{1+PC} & \frac{-C}{1+PC} & \frac{CF}{1+PC}
\end{bmatrix}
\begin{bmatrix}
d \\
n \\
r
\end{bmatrix}
\]

Frequency Domain Specifications

Specifications on the open loop transfer function \( (L) \)
- Gain crossover frequency, \( \omega_{gc} \), is the lowest frequency at which loop gain = 1
- Gain margin, \( gm \), is the amount the loop gain can be increased before instability
- Phase margin, \( \phi_m \), is amount of phase lag required to generate instability

Specifications on closed loop frequency response (eg \( H_{yr}, H_{yd}, \text{etc} \))
- Resonant peak, \( Mr \), is the largest value of the frequency response
- Peak frequency, \( \omega_p \), is the frequency where the maximum occurs
- Bandwidth, \( \omega_b \), is the frequency where the gain has decreased to \( 1/\sqrt{2} \)

Basic idea: convert specs on closed loop to specs on open loop
- Bandwidth \( \approx \) value for which \( |L| = 1 \)
- Resonant peak set by phase margin
- Keep \( L \) large to set \( H_{yr} \approx 1 \)
Time Domain Specs $\rightarrow$ Frequency Domain Specs

**Time domain specifications**

Map to frequency domain for second order system

\[
L(s) = \frac{k}{s^2 + bs} \quad \text{and} \quad H_{gy} = \frac{k}{s^2 + bs + k}
\]

- Use properties of second order systems (Ch 7)
- HW #8, problem 1 (CDS 110 only)

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"Loop Shaping": Design Loop Transfer Function

Translate specs to "loop shape"

\[
L(s) = P(s)C(s)
\]

- Design C(s) to obey constraints

Typical loop constraints

- High gain at low frequency
  - Good tracking, disturbance rejection at low freqs
- Low gain at high frequency
  - Avoid amplifying noise
- Sufficiently high bandwidth
  - Good rise/settling time
- Shallow slope at crossover
  - Sufficient phase margin for robustness, low overshoot

Key constraint: slope of gain curve determines phase curve

- Can’t independently adjust
- Eg: slope at crossover sets PM
**Loop Shaping: Basic Approach**

- **Disturbance rejection**
  
  \[ H_{ed} = \frac{-P}{1 + L} \]

  - Would like \( H_{ed} \) to be small make \( \Rightarrow \) large \( L(s) \)
  - Typically require this in low frequency range

- **High frequency measurement noise**
  
  \[ H_{un} = \frac{-L}{P(1 + L)} \]

  - Want to make sure that \( H_{un} \) is small (avoid amplifying noise) \( \Rightarrow \) small \( L(s) \)
  - Typically generates constraints in high frequency range

- **Robustness: gain and phase margin**
  - Focus on gain crossover region: make sure the slope is “gentle” at gain crossover
  - Fundamental tradeoff: transition from high gain to low gain through crossover

**Design Method #1: Process Inversion**

- **Simple trick: invert out process**
  
  - Write all performance specs in terms of the desired loop transfer function
  - Choose \( L(s) \) that satisfies specifications
  - Choose controller by inverting \( P(s) \)

  \[ C(s) = \frac{L(s)}{P(s)} \]

- **Pros**
  - Very easy design process
  - \( L(s) = k/s \) often works very well
  - Can be used as a first cut, with additional shaping to tune design

- **Cons**
  - High order controllers (at least same order as the process you are controlling)
  - Requires “perfect” model of your process (since you are inverting it)
  - Can generate non-proper controllers (order(num) > order(den))
    - Difficult to implement, plus amplifies noise at high frequency (\( C(\infty) = \infty \))
    - Fix by adding high frequency poles to roll off control response at high frequency
  - Does not work if you have right half plane poles or zeros (get internal instability)
Design Method #2: Lead compensation

Use to increase phase in frequency band
- Effect: lifts phase by increasing gain at high frequency
- Very useful controller; increases PM
- Bode: add phase between zero and pole
- Nyquist: increase phase margin

Example: Lead Compensation for Second Order System

System description
\[ P(s) = \frac{p_1 p_2}{(s + p_1)(s + p_2)} \]
- Poles: \( p_1 = 1 \), \( p_2 = 5 \)

Control specs
- Track constant reference with error < 1%
- Good tracking up to 100 rad/s (less than 10% error)
- Overshoot less than 10%  
  - Gives PM of \( \sim 60 \) deg

Try a lead compensator
\[ C(s) = K \frac{s + a}{s + b} \]
- Want gain crossover at approximately 100 rad/sec => center phase gain there
- Set zero frequency gain of controller to give small error => \( |L(0)| > 100 \)
- \( a = 20 \), \( b = 500 \), \( K = 10,000 \) (gives \( |C(0)| = |L(0)| = 400 \))
Safety Check: Nyquist + Gang of 4

Nyquist verifies closed loop stability
- Infinite GM; good phase margin

Gang of 4 shows high noise sens’y
- Factor of 10,000 gain at high freq
- Step responses show similar sensitivity

Solution? (HW #8…)

Summary: Loop Shaping

Loop Shaping for Stability & Performance
- Steady state error, bandwidth, tracking response
- Specs can be on any input/output response pair

Main ideas
- Performance specs give bounds on loop transfer function
- Use controller to shape response
- Gain/phase relationships constrain design approach
- Standard compensators: proportional, lead, PI

Things to remember (for homework and exams)
- Always plot Nyquist to verify stability/robustness
- Check gang of 4 to make sure that noise and disturbance responses also look OK