1 Time-Delay Systems

1.1 Introduction

In control systems a challenging area is operating in the presence of delays. Delays can be attributed to acquiring information to make a decision, creating a control decision and/or executing the decisions. For example in the control system of an aircraft delays can be cause by actuators and sensors. This is very common in computer controlled systems.

1.2 Mathematical Formulation

Introducing delay to the system shifts us into a different class of differential equations than those we have been dealing with: ordinary differential equations to functional differential equations. The simplest system with a time delay has the form:

$$\dot{x}(t) = -x(t-h),\, x(t) \in \mathbf{R}$$

where $h > 0$ is the time-delay.

For a system $y(t) = x(t-\tau)$, where τ is the time delay, the transfer function for such a system $\dfrac{Y(s)}{X(s)} = e^{-s\tau}$.

1.3 Block Diagram

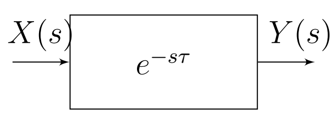

Figure 1: Simple Time Delay Block Diagram

where $Y(s) = e^{-s\tau} * X(s)$

1.4 Bode Plot

$$| \, e^{-i\omega\tau} \, | = 1$$

$$\angle e^{-i\omega t} = \omega\tau$$

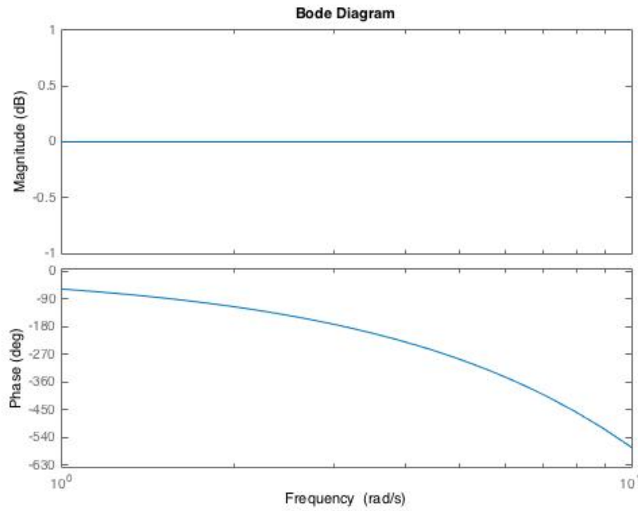

Figure 2: Bode Plot of Time Delay

1. On the complex plane, the polar plot of $e^{-i\omega\tau}$ corresponds to a circle of unit radius. The Bode magnitude plot is therefore a horizontal line at 1, or 0 dB.
2. The phase is usually wrapped to $(2\pi, 0]$ in Bode plots, so the Bode phase plot of a time delay is usually portrayed as in figure 2
3. Limits gain margin
4. Creates large phase lag at high frequencies

1.5 Padé Approximation

The transfer function of a time-delay is irrational. In some situations as in frequency response based analysis of control systems containing a time-delay it is necessary to substitute the time delay with an approximation in form of a rational transfer function. The most common approximation is the Padé approximation:

$$e^{-s\tau} \approx \frac{1 - \frac{s\tau}{2}}{1 + \frac{s\tau}{2}}$$

2 Nyquist Plot

2.1 Introduction

Recall the loop transfer function of a system is

$$L(s) = P(s)C(s)$$

The frequency response of the loop transfer function can be represented by plotting the complex number $L(i\omega)$ as a function of ω. Such a plot is called the Nyquist plot and the curve is called the Nyquist curve.The magnitude $\mid L(i\omega) \mid$ is called the loop gain because it tells how much the signal is amplified as is passes around the feedback loop and the $\angle L(i\omega)$ is called the phase.

2.2 Nyquist Stability Test

The Nyquist plot allows us to predict the stability and performance of a closed-loop system by observing its open-loop behavior. The closed-loop system is stable if and only if the net number of clockwise encirclement of the points $s = -1 + j0$ by the Nyquist diagram of $L(s)$ plus the number of poles of $L(s)$ in the right half-plane is zero.

In other words: The number of unstable poles of the closed loop system is given by the number of open loop unstable poles plus the number of clockwise encirclements of the point $s = 1$. Let N be the net number of clockwise encirclements of 1 by $L(s)$ when s encircles the Nyquist contour Γ in the clockwise direction. The closed loop system then has $Z = N + P$ poles in the right half-plane. We want $Z = 0$ for close loop stability.

2.3 Basic Example

Consider the transfer function

$$L(s) = \frac{1}{\frac{s}{10} + 1}$$

The bode plot for this transfer function is:

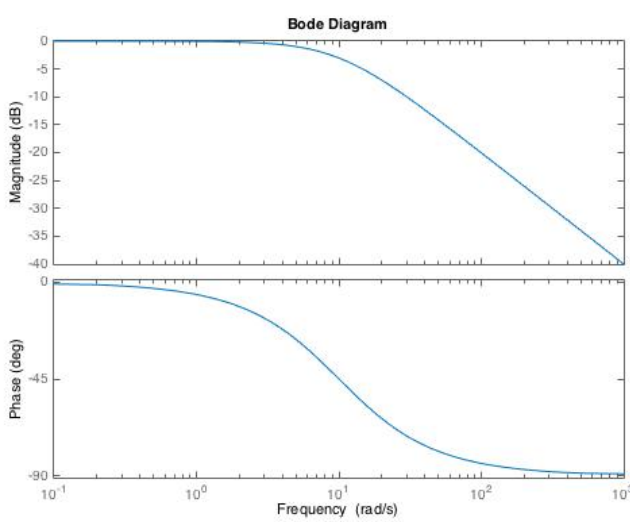

Figure 3: Bode Plot

Considering the critical points in the table below:

ω	Gain (dB)	Phase
0	0	0
10	-3	$-45°$
infinity	infinity	$-90°$

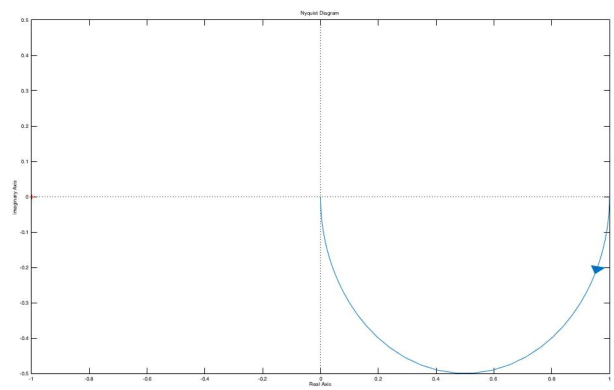

Figure 4: Nyquist Contour

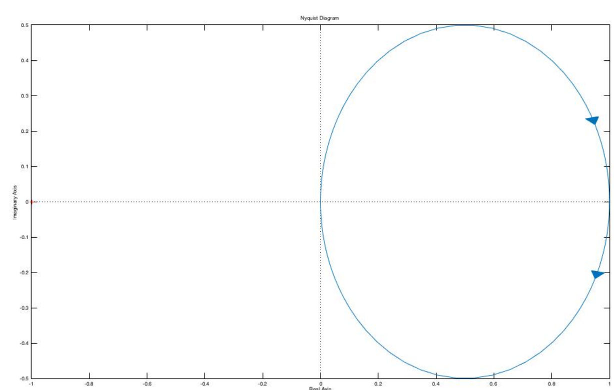

Figure 5: Nyquist Plot

Note: Matlab nyquist command does not take poles or zeros on the $j\omega$ axis into account and therefore use the nyquist function provided on the course website if the system has poles zeros on the $j\omega$ axis.

2.4 Example with Delay

Note: The only tool that would always provide you with correct answers in all cases when a time delay is present is the Nyquist plot.

Consider the system p:

$$L(s) = \frac{s+3}{s^2 + 0.3s + 1}$$

This yields the following:

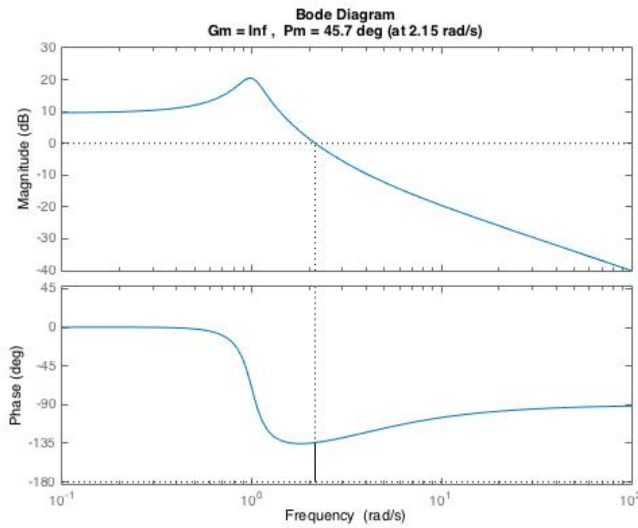

Figure 6: Bode Plot with Margins

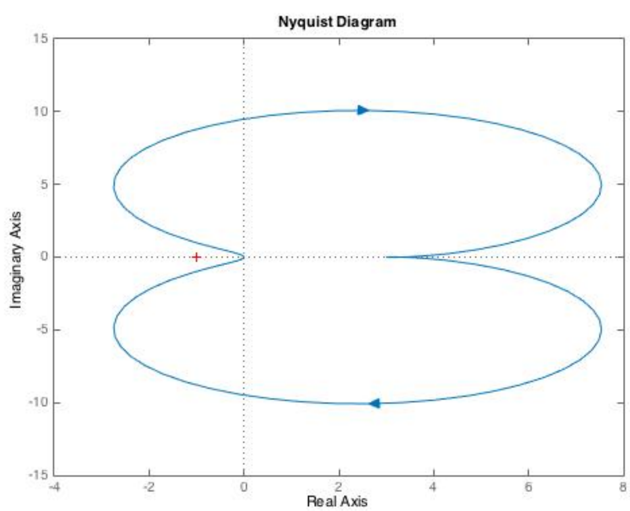

Figure 7: Nyquist Plot

Recall: Let ω_{pc} represent the phase crossover frequency. Then the gain margin for the system is given by

$$g_m = \frac{1}{\mid L(i\omega_{pc}) \mid}$$

Let ω_{gc} be the gain crossover frequency, the smallest frequency where the loop transfer function $L(s)$ has unit magnitude. Then for a system with monotonically decreasing gain, the phase margin is given by

$$\phi_m = \pi + \arg\, L(i\omega_{gc})$$

Lets add a time delay of 1 second:

$$L(s) = \frac{s+3}{s^2 + 0.3s + 1} e^{-s}$$

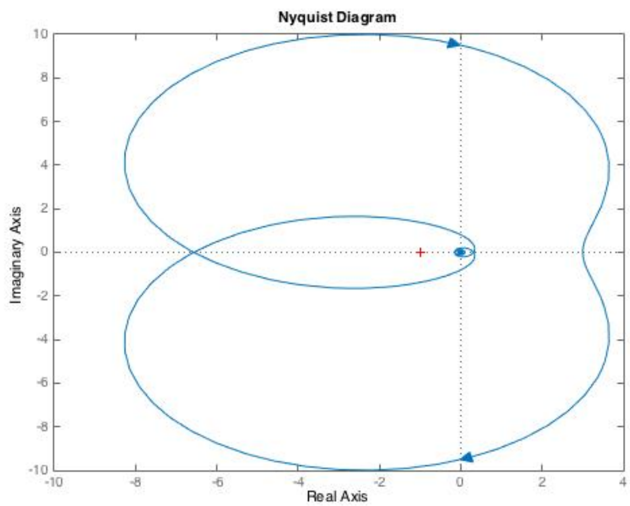

Figure 8: Nyquist Plot with Delay