CDS 101/110a: Lecture 6-1
Transfer Functions

Richard M. Murray
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Goals:

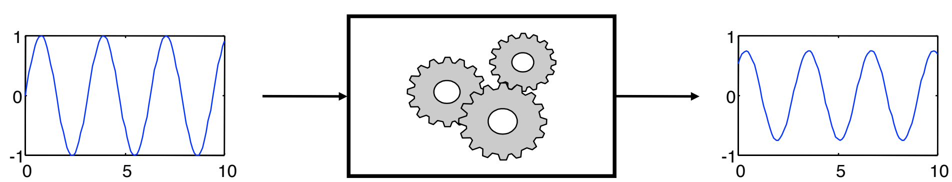
- Define the input/output transfer function of a linear system
- Describe how to use Bode plots to understand the frequency response
- Understand the relationships between frequency response, transfer function, and state-space model
- Introduce block diagram algebra for computing transfer functions of interconnected systems

Reading:

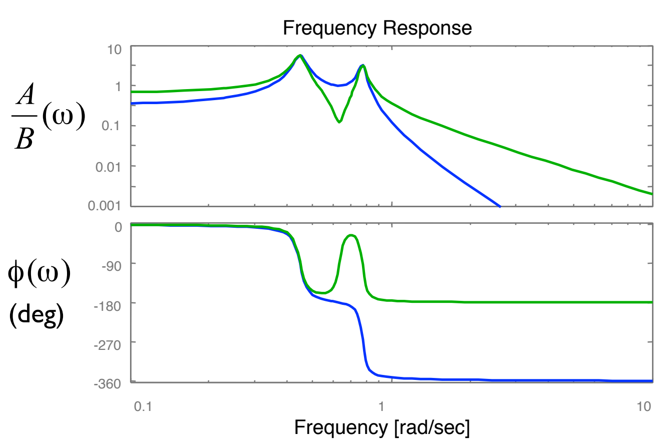
- Åström and Murray, Feedback Systems, Ch 9

Frequency Domain Modeling

Defn. The frequency response of a linear system is the relationship between the gain and phase of a sinusoidal input and the corresponding steady state (sinusoidal) output.



$u = A\sin(\omega t)$ $\qquad$ $y = B\sin(\omega t + \phi)$



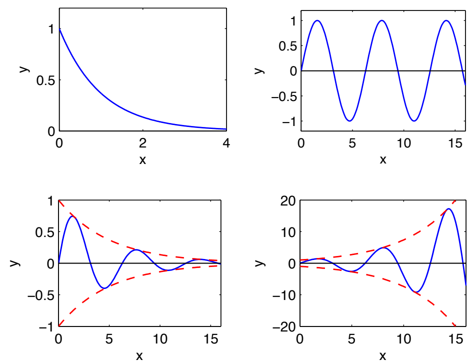
Bode plot (1940; Henrik Bode)

- Plot gain and phase vs input frequency
- Gain is plotting using log-log plot
- Phase is plotting with log-linear plot
- Can read off the system response to a sinusoid – in the lab or in simulations
- Linearity ⇒ can construct response to any input (via Fourier decomposition)
- Key idea: do all computations in terms of gain and phase (frequency domain)

Transmission of Exponential Signals

Exponential signal: $e^{st} = e^{(\sigma + i\omega)t} = e^{\sigma t} e^{i\omega t} = e^{\sigma t}(\cos \omega t + i \sin \omega t)$

- Construct constant inputs + sines/cosines by linear combinations
  - Constant: $u(t) = c = ce^{0t}$
  - Sinusoid: $u(t) = A\sin(\omega t) = \dfrac{A}{2i}(e^{i\omega t} - e^{-i\omega t})$
  - Decaying sinusoid: $u(t) = Ae^{-\sigma t}\sin(\omega t)$
- Exponential response can be computed via the convolution equation



$$
\begin{aligned}
x(t) &= e^{At}x(0) + \int_0^t e^{A(t-\tau)} B e^{s\tau}\, d\tau \\
&= e^{At}x(0) + e^{At}(sI - A)^{-1} e^{(sI-A)\tau}\Big|_{\tau=0}^{t} B \\
&= e^{At}x(0) + e^{At}(sI - A)^{-1}\Big(e^{(sI-A)t} - I\Big)B \\
&= e^{At}\Big(x(0) - (sI - A)^{-1}B\Big) + (sI - A)^{-1}Be^{st}
\end{aligned}
$$

$$
\begin{aligned}
y(t) &= Cx(t) + Du(t) \\
&= Ce^{At}\Big(x(0) - (sI - A)^{-1}B\Big) + \Big(C(sI - A)^{-1}B + D\Big)e^{st}
\end{aligned}
$$

Transfer Function and Frequency Response

Exponential response of a linear state space system

$$y(t) = Ce^{At}\Big(x(0) - (sI - A)^{-1}B\Big) + \Big(C(sI - A)^{-1}B + D\Big)e^{st}$$

transient steady state

Transfer function

- Steady state response is proportional to exponential input => look at input/output ratio
- $G(s) = C(sI - A)^{-1}B + D$ is the transfer function between input and output

Common transfer functions

$\dot{y} = u$	$\frac{1}{s}$
$y = \dot{u}$	s
$\dot{y} + ay = u$	$\frac{1}{s+a}$
$\ddot{y} = u$	$\frac{1}{s^2}$
$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = u$	$\frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2}$
$y = k_p u + k_d \dot{u} + k_i \int u$	$k_p + k_d s + \frac{k_i}{s}$
$y(t) = u(t - \tau)$	$e^{-\tau s}$

Frequency response

$$
\begin{aligned}
u(t) &= A\sin\omega t = \frac{A}{2i}(e^{i\omega t} - e^{-i\omega t}) \\
y_{ss}(t) &= \frac{A}{2i}\Big(G(i\omega)e^{i\omega t} - G(-i\omega)e^{-i\omega t}\Big) \\
&= A \cdot |G(i\omega)| \sin(\omega t + \arg G(i\omega))
\end{aligned}
$$

gain phase

Transfer Function Properties

Thm. The transfer function for a linear system Σ=(A,B,C,D) is given by

$$G(s) = C(sI - A)^{-1}B + D \qquad s \in \mathbb{C}$$

Thm. The transfer function G(s) corresponding to Σ=(A,B,C,D) has the following properties:

- H(s) is a ratio of polynomials n(s)/d(s) where d(s) is the characteristic equation for the matrix A and n(s) has order less than or equal to d(s).
- The steady state frequency response of Σ has gain |G(jω)| and phase arg G(jω):

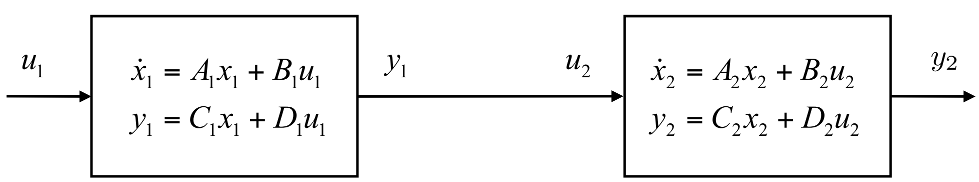
$$\begin{aligned} u &= A\sin(\omega t) \\ y &= |G(i\omega)| A \sin(\omega t + \arg G(i\omega)) + \text{transients} \end{aligned}$$

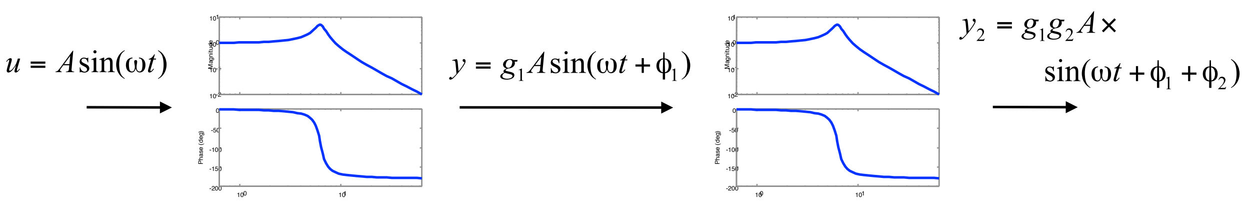
Remarks

- Formally, can show that G(s) is the Laplace transform of the impulse response of Σ
- Often write "y = G(s)u" for Y(s) = G(s)U(s), where Y(s) & U(s) are Laplace transforms of y(t) and u(t). (Multiplication in Laplace domain corresponds to convolution.)
- MATLAB: G = ss2tf(A, B, C, D)

Series Interconnections

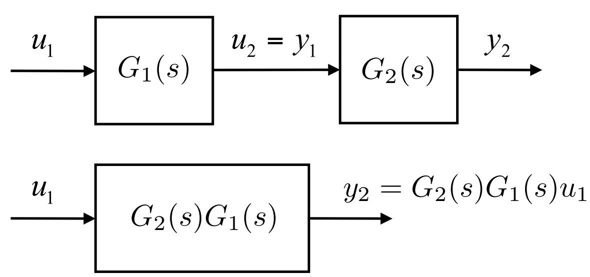
Q: what happens when we connect two systems together in series?



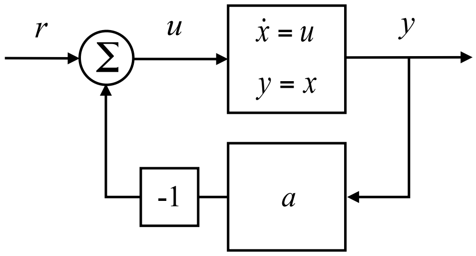


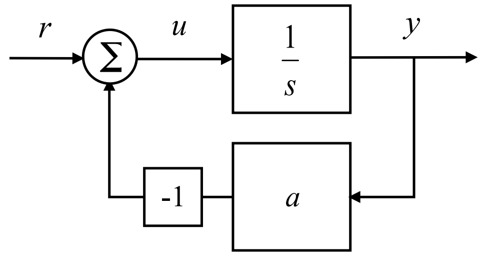
A: Transfer functions multiply

- Gains multiply, phases add
- Generally: transfer functions well formulated for frequency domain interconnections
  - Convolution → multiplication
- MATLAB/python: G = series(G1, G2)



Feedback Interconnection





State space derivation

$$\dot{x} = u = r - ay = -ax + r$$
$$y = x$$

Frequency response $\quad r = A\sin(\omega t)$

$$y = \left|\frac{1}{\sqrt{a^2+\omega^2}}\right| \sin\left(\omega t - \tan^{-1}\left(\frac{\omega}{a}\right)\right)$$

Transfer function derivation

$$y = \frac{u}{s} = \frac{r - ay}{s}$$
$$y = \frac{r}{s+a} = G(s)r$$

Frequency response

$$y = |G(i\omega)| \sin(\omega t + \angle G(i\omega))$$

- MATLAB: G = feedback(sys, a) $\leftarrow$ works for either state space or transfer functions

Poles and Zeros

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

$$G(s) = \frac{n(s)}{d(s)}$$
$$d(s) = \det(sI - A)$$

- Roots of *d*(*s*) are called *poles* of *G*(*s*)
- Roots of *n*(*s*) are called *zeros* of *G*(*s*)

Poles of G(s) determine the stability of the (closed loop) system

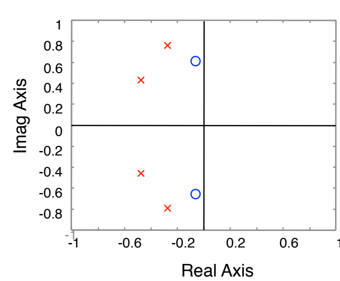
- Denominator of transfer function = characteristic polynomial of state space system
- Provides easy method for computing stability of systems
- Right half plane (RHP) poles (Re > 0) correspond to unstable systems

Zeros of G(s) related to frequency ranges with limited transmission

- A pure imaginary zero at s = iω blocks any output at that frequency (G(iω) = 0)
- Zeros provide limits on performance, especially RHP zeros (more on this later)

MATLAB: pole(G), zero (G), pzmap(G)

$$G(s) = k\frac{s^2 + b_1 s + b_2}{s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4} \xrightarrow{\text{pzmap(G)}}$$



Sketching the Bode Plot for a Transfer Function (1/2)

Evaluate transfer function on imaginary axis

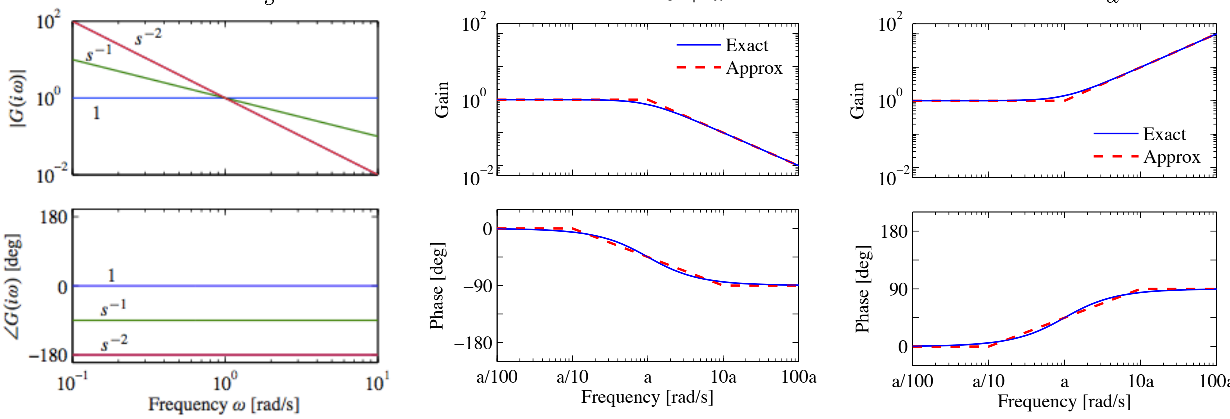
$$M = |G(i\omega)|, \qquad \varphi = \arctan \frac{\operatorname{Im} G(i\omega)}{\operatorname{Re} G(i\omega)}$$

$$\log |G(i\omega)| \approx \begin{cases} 0 & \text{if } \omega < a \\ \log a - \log \omega & \text{if } \omega > a, \end{cases}$$

- Plot gain (M) on log/log scale
- Plot phase (φ) on log/linear scale
- Piecewise linear approximations available

$$\angle G(i\omega) \approx \begin{cases} 0 & \text{if } \omega < a/10 \\ -45 - 45(\log \omega - \log a) & a/10 < \omega < 10a \\ -90 & \text{if } \omega > 10a. \end{cases}$$

$$G(s) = \frac{1}{s^k} \qquad G(s) = \frac{a}{s+a} \qquad G(s) = \frac{s+a}{a}$$



Sketching the Bode Plot for a Transfer Function (2/2)

Complex poles $G(s) = \dfrac{\omega_0^2}{s^2 + 2\omega_0\zeta s + \omega_0^2}$

$$\log |G(i\omega)| \approx \begin{cases} 0 & \text{if } \omega \ll \omega_0 \\ 2\log \omega_0 - 2\log \omega & \text{if } \omega \gg \omega_0, \end{cases}$$

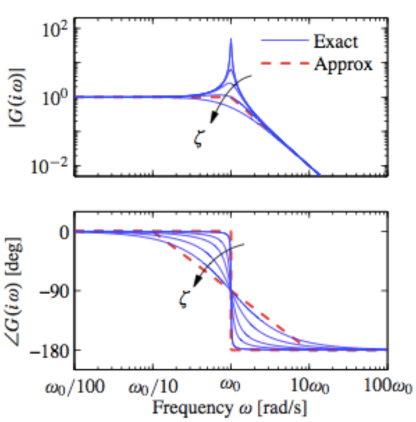
$$\angle G(i\omega) \approx \begin{cases} 0 & \text{if } \omega \ll \omega_0 \\ -180 & \text{if } \omega \gg \omega_0. \end{cases}$$

Ratios of products $G(s) = \dfrac{b_1(s) b_2(s)}{a_1(s) a_2(s)}$

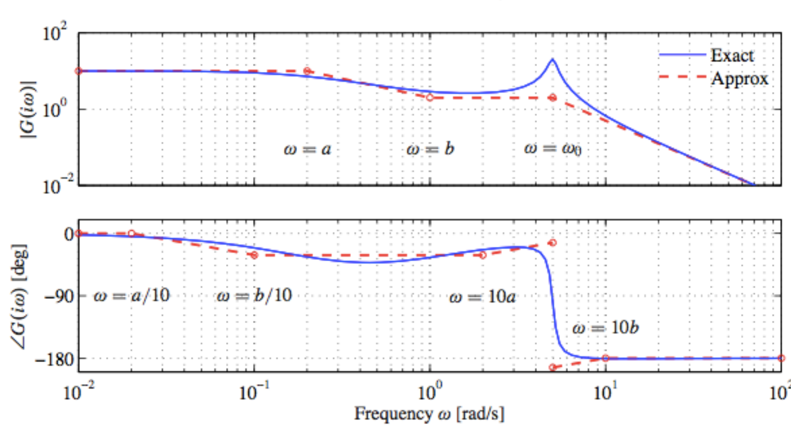
$$\log |G(s)| = \log |b_1(s)| + \log |b_2(s)| - \log |a_1(s)| - \log |a_2(s)|$$

$$\angle G(s) = \angle b_1(s) + \angle b_2(s) - \angle a_1(s) - \angle a_2(s)$$

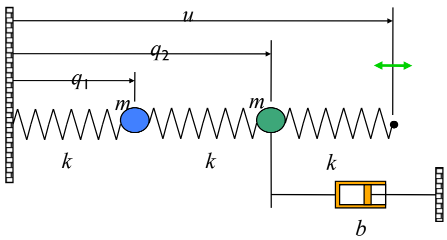
$$G(s) = \frac{\omega_0^2}{s^2 + 2\omega_0\zeta s + \omega_0^2}$$



$$G(s) = \frac{k(s+b)}{(s+a)(s^2 + 2\zeta\omega_0 s + \omega_0^2)}, \qquad a \ll b \ll \omega_0.$$



Example: Coupled Masses



$$H_{q_1 f} = \frac{0.04}{s^4 + 0.08s^3 + 0.8016s^2 + 0.032s + 0.12}$$

$$H_{q_2 f} = \frac{0.2s^2 + 0.008s + 0.08}{s^4 + 0.08s^3 + 0.8016s^2 + 0.032s + 0.12}$$

Poles (Hq1f and Hq2f)

- -0.0200 ± 0.7743j
- -0.0200 ± 0.4468j

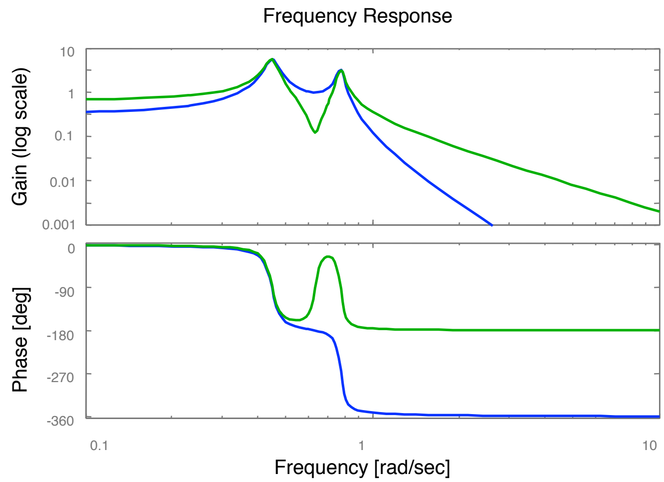
Zeros (Hq2f)

- -0.0200 ± 0.6321j

Interpretation

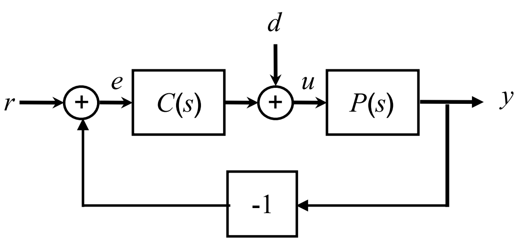
- Zeros in Hq2f give low response at $\omega \approx 0.6321$

MATLAB: bode(H)



Block Diagram Algebra

Basic idea: treat transfer functions as multiplication, write down equations



$$y = P(s)u$$
$$u = d + C(s)e$$
$$e = r - y$$

Manipulate equations to compute desired signals

$$\begin{aligned} e &= r - y \\ &= r - P(s)u \\ &= r - P(s)(d + C(s)e) \end{aligned}$$

$$(1 + P(s)C(s))e = r - P(s)d$$

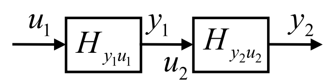
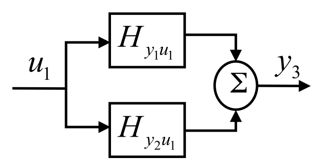
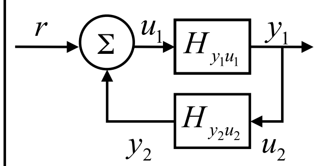
$$e = \underbrace{\frac{1}{1 + P(s)C(s)}}_{H_{er}} r - \underbrace{\frac{P(s)}{1 + P(s)C(s)}}_{H_{ed}} d$$

Note: linearity gives superposition of terms

Algebra works because we are working in frequency domain

- Time domain (ODE) representations are not as easy to work with
- Formally, all of this works because of Laplace transforms (see text)

Block Diagram Algebra

Type	Diagram	Transfer function
Series		$H_{y_2u_1} = H_{y_2u_2}H_{y_1u_1} = \dfrac{n_1n_2}{d_1d_2}$
Parallel		$H_{y_3u_1} = H_{y_2u_1} + H_{y_1u_1} = \dfrac{n_1d_2 + n_2d_1}{d_1d_2}$
Feedback		$H_{y_1r} = \dfrac{H_{y_1u_1}}{1 + H_{y_1u_1}H_{y_2u_2}} = \dfrac{n_1d_2}{n_1n_2 + d_1d_2}$

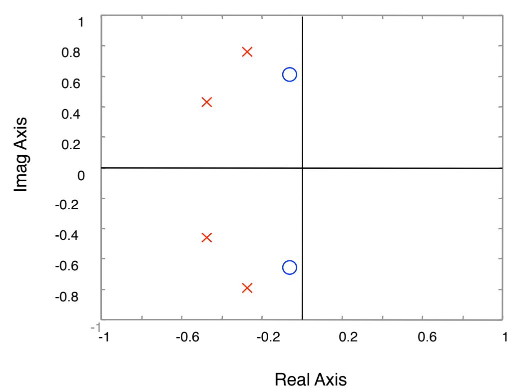
- These are the basic manipulations needed; some others are possible
- Formally, could work all of this out using the original ODEs (⇒ nothing really new)

MATLAB/Python manipulation of transfer functions

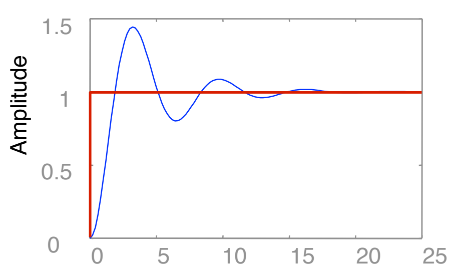
Creating transfer functions
- [num, den] = ss2tf(A, B, C, D)
- sys = tf(num, den)
- num, den = [1 a b] → s2 + as + b

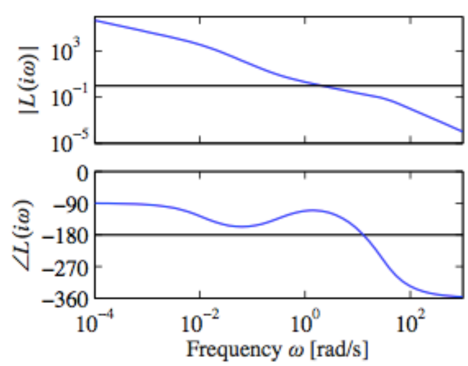
Interconnecting blocks
- sys= series(sys1, sys2), parallel, feedback

Computing poles and zeros
- pole(sys), zero(sys)
- pzmap(sys)

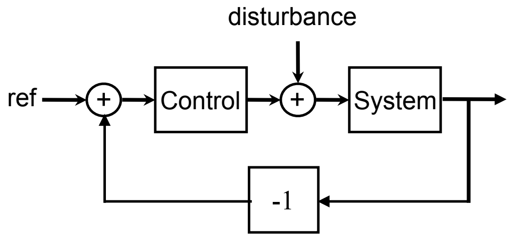


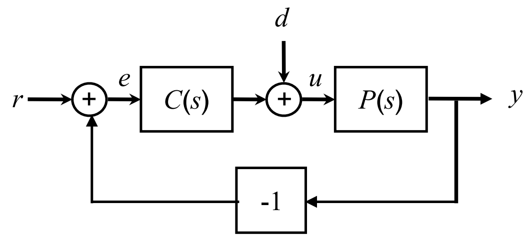
I/O response
- initial(sys)
- step(sys)
- lsim(sys)
- bode(sys)





Control Analysis and Design Using Transfer Functions





Transfer functions provide a method for "block diagram algebra"

- Easy to compute transfer functions between various inputs and outputs
  - Her(s) is the transfer function between the reference and the error
  - Hed(s) is the transfer function between the disturbance and the error

Transfer functions provide a method for performance specification

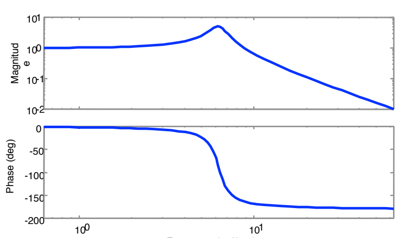
- Since transfer functions provide frequency response directly, it is convenient to work in the "frequency domain"
  - Her(s) should be small in the frequency range 0 to 10 Hz (good tracking)

Key idea: perform all analysis and design for linear systems in "frequency domain"

- Convert specifications on time response to specifications on frequency response
- "Shape" the frequency response by design of controller transfer function (Ch 10-13)

Summary: Frequency Response & Transfer Functions

$u = A\sin(\omega t)$ → $\dot{x} = Ax + Bu$, $y = Cx + Du$, $x(0) = 0$ → $y_{ss} = A \cdot |G(i\omega)| \times \sin(\omega t + \arg G(i\omega))$



$$G(s) = C(sI - A)^{-1}B + D$$

$$G_{y_2u_1} = G_{y_2u_2}G_{y_1u_1} = \frac{n_1n_2}{d_1d_2}$$