CDS 101/110: Lecture 3.1
Linear Systems

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Goals for Today:

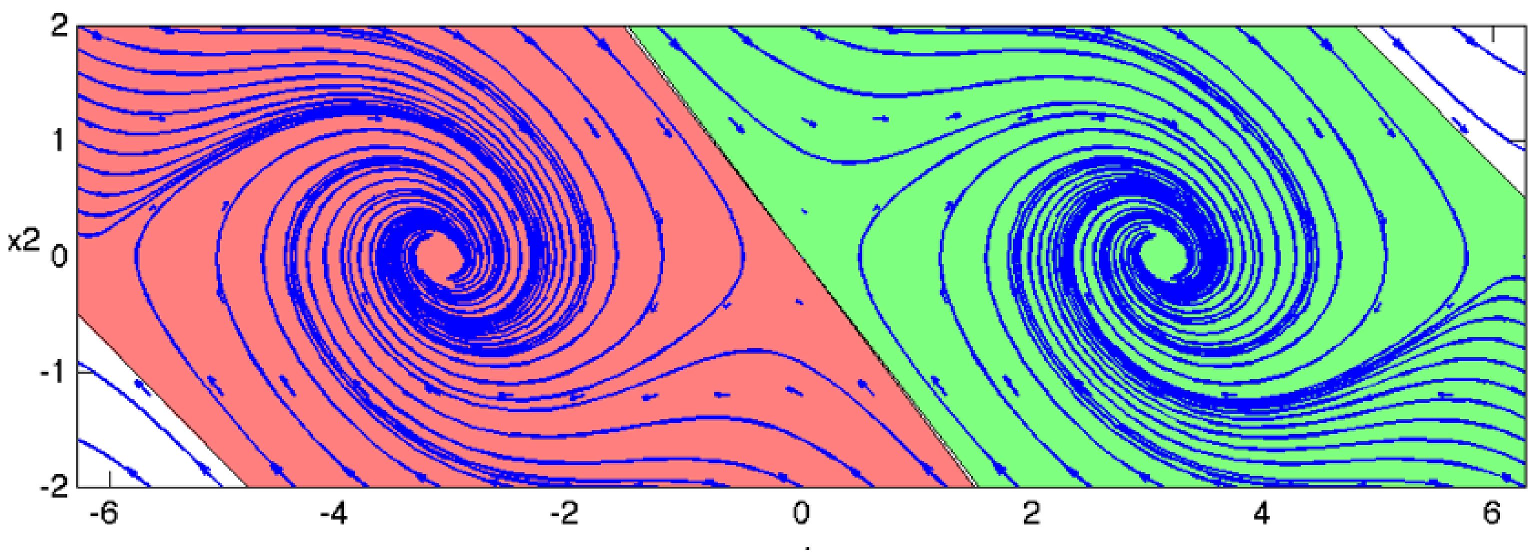
- Describe and motivate linear system models:
- Summarize properties, examples, and tools
  - Convolution equation describing solution in response to an input
  - Step response, impulse response
  - Frequency response
- Characterize stability and performance of linear systems in terms of eigenvalues

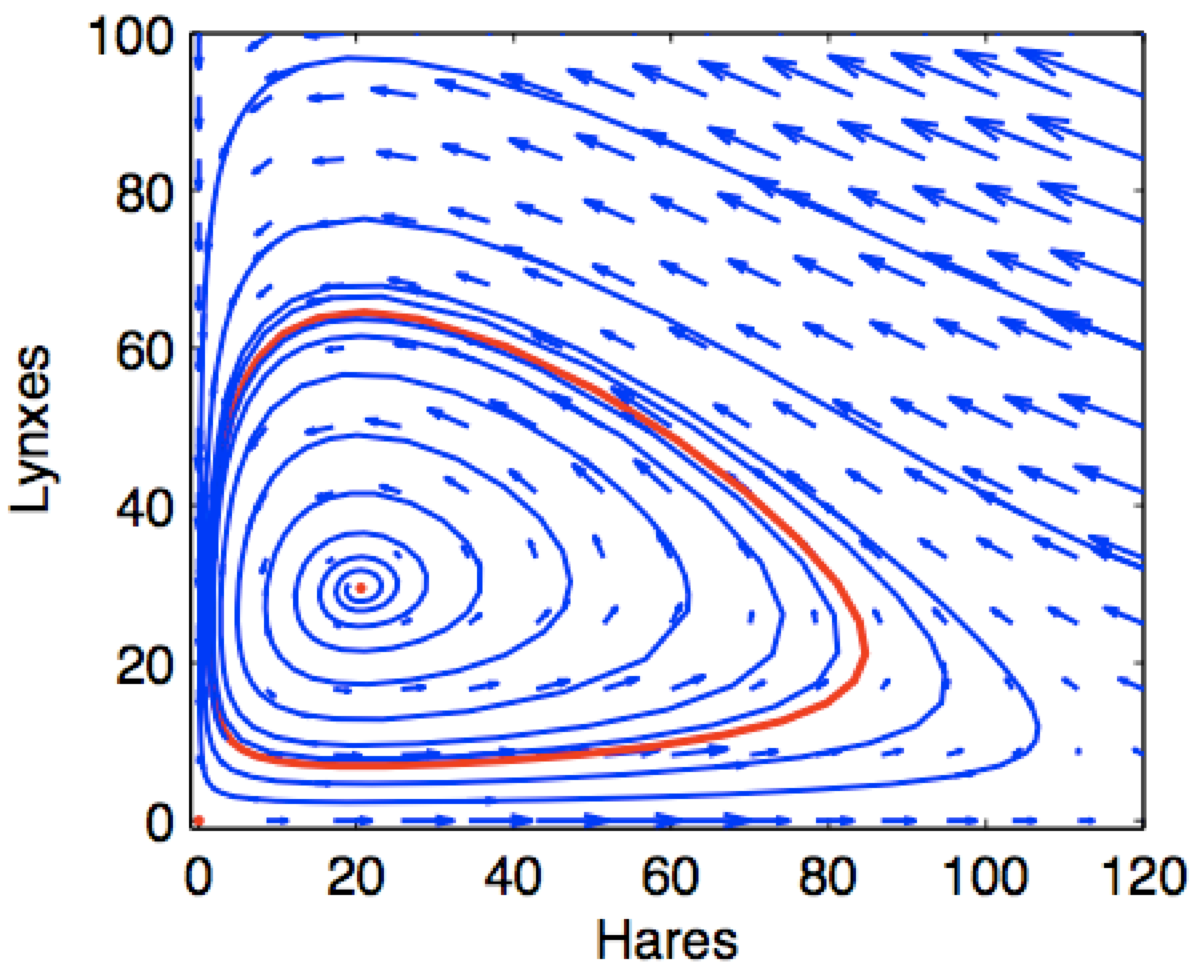
Reading:

- Åström and Murray, Analysis and Design of Feedback Systems, Ch 5

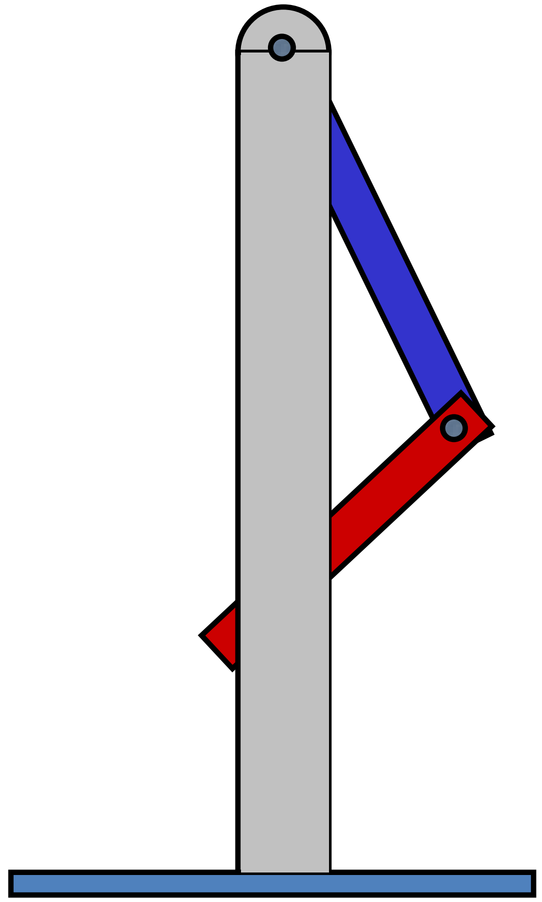
Homework: On course website Monday night

Quick Review: Stability and Performance





- Key topics
  - Stability of equilibrium points
  - Eigenvalues determine stability for linear systems
  - Local versus global behavior



CDS 140, 240 goes into much more detail

Linear Systems

Recall: Linearity of Functions $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$

- Addition: $f(x + y) = f(x) + f(y)$
- Scaling: $f(\alpha x) = \alpha f(x)$
- Zero at the Origin: $f(0) = 0$

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$$

Linear *System:* $S: u(t) \rightarrow x(t)$

- If $S: u_1(t) \rightarrow x_1(t)$; $S: u_2(t) \rightarrow x_2(t)$
  - $\alpha x_1(t) + \beta x_2(t) = S\{\alpha u_1(t) + \beta u_2(t)\}$

Linear *Control System:*

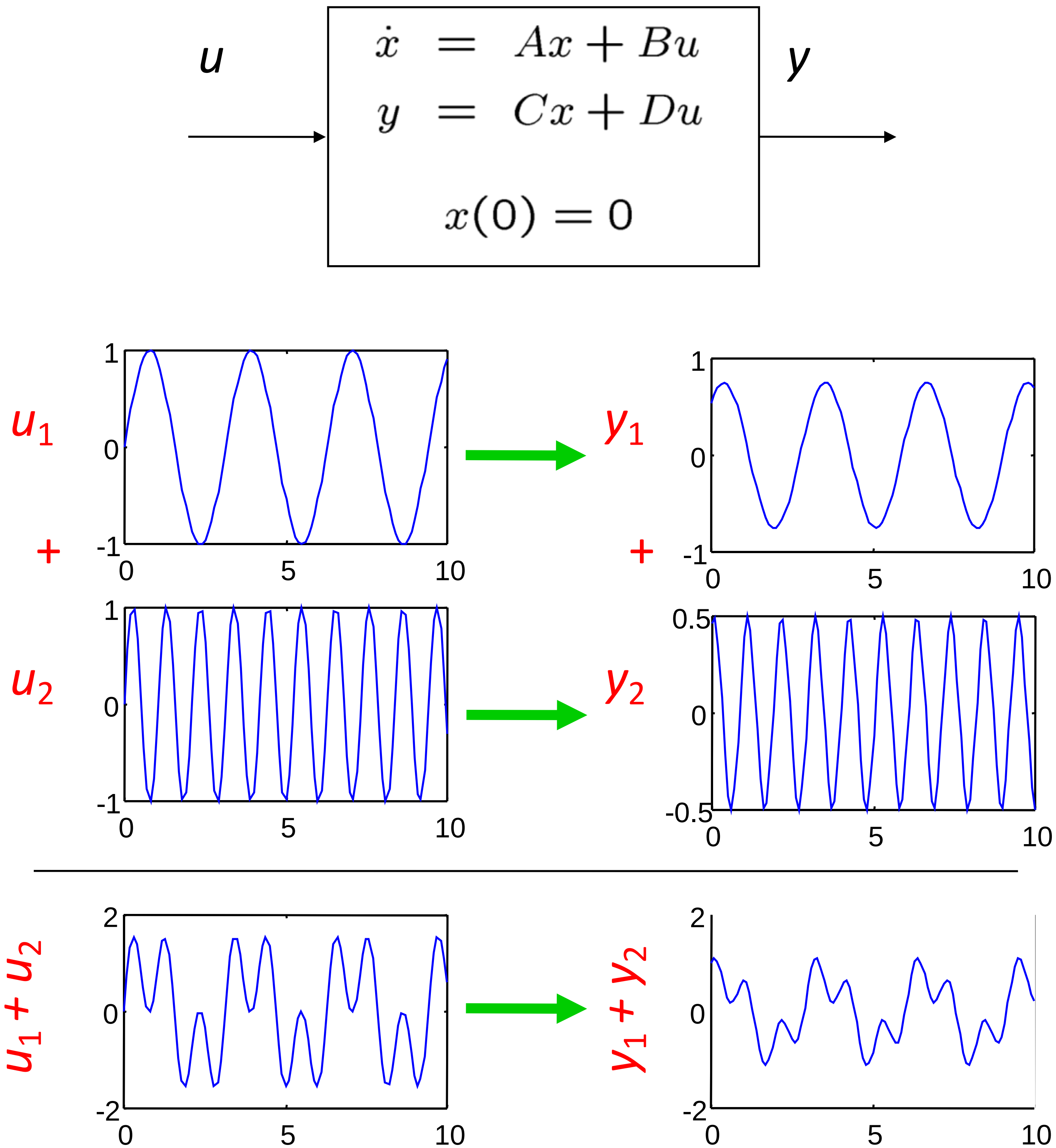
- $\dot{x}(t) = A\, x(t) + B\, u(t)$
- $y(t) = C\, x(t) + D\, u(t)$

$x(t)$ is system "state";

$u(t)$ are control inputs

$y(t)$ is the system output, (what is observed)

Linear Systems

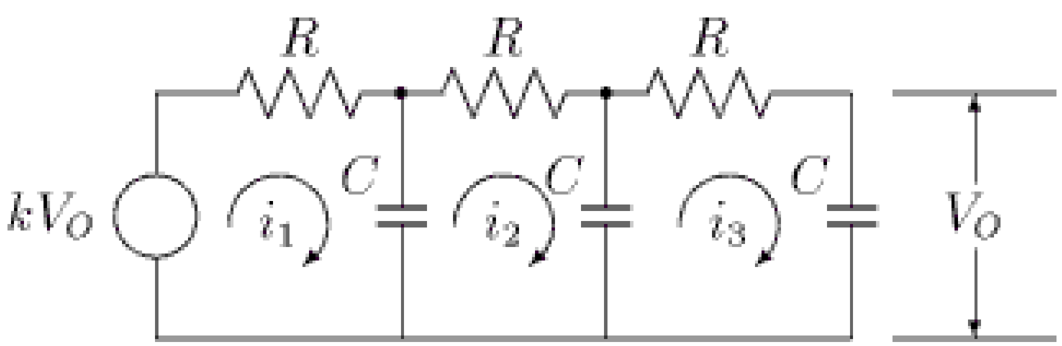


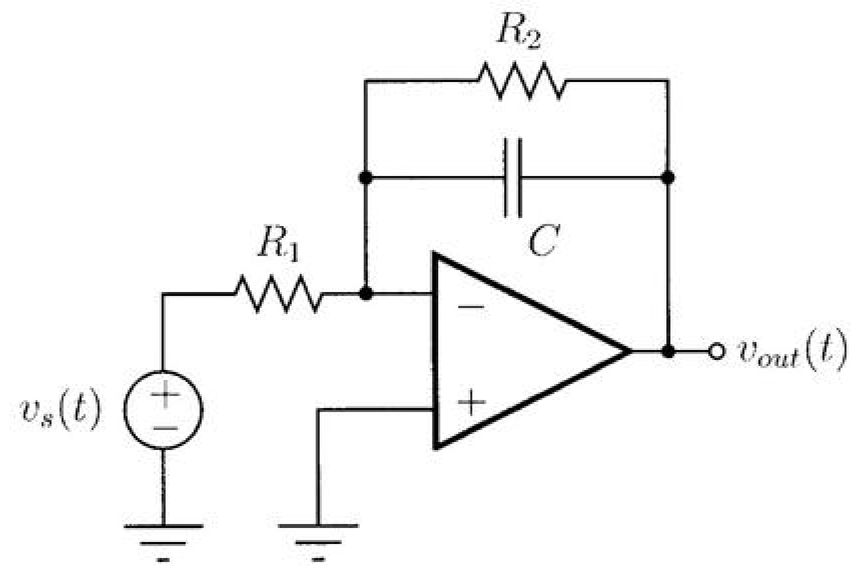
- Input/output linearity at x(0) = 0
  - Linear systems are linear in initial condition and input $\Rightarrow$ need to use x(0) = 0 to add outputs together
  - For different initial conditions, you need to be more careful
- Linear system $\Rightarrow$ step response and frequency response scale with input amplitude
  - 2X input $\Rightarrow$ 2X output
  - Allows us to use ratios and percentages in step or frequency response. *These are independent of input amplitude*
  - Limitation: input saturation $\Rightarrow$ only holds up to certain input amplitude

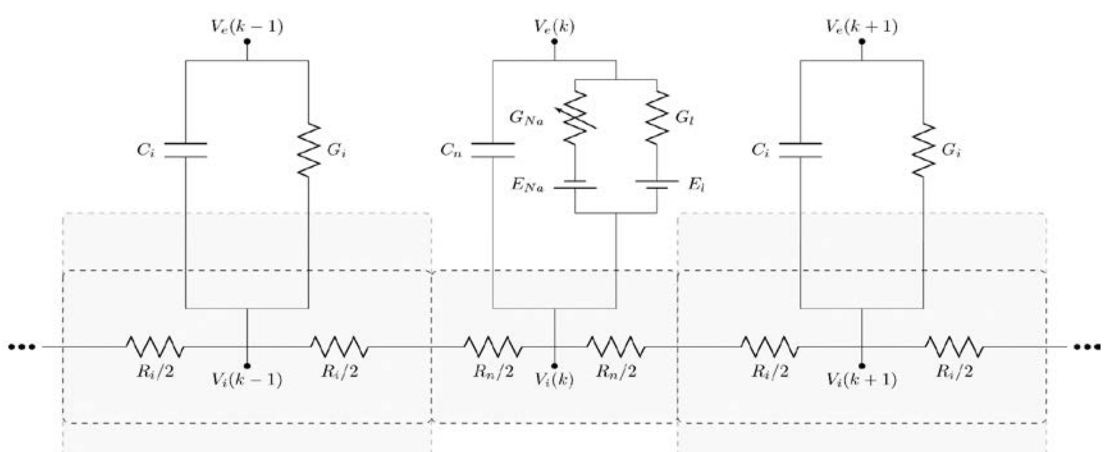
Why are Linear Systems Important?

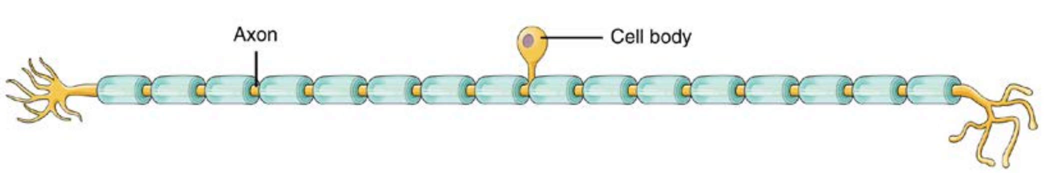
Many important *examples*

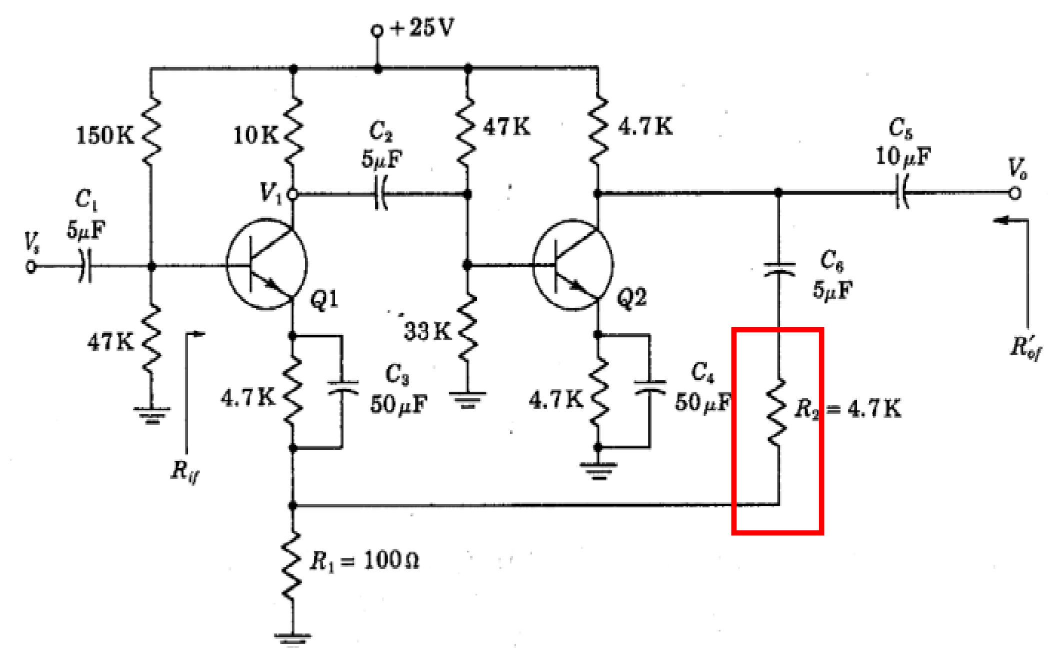
- Electronic circuits









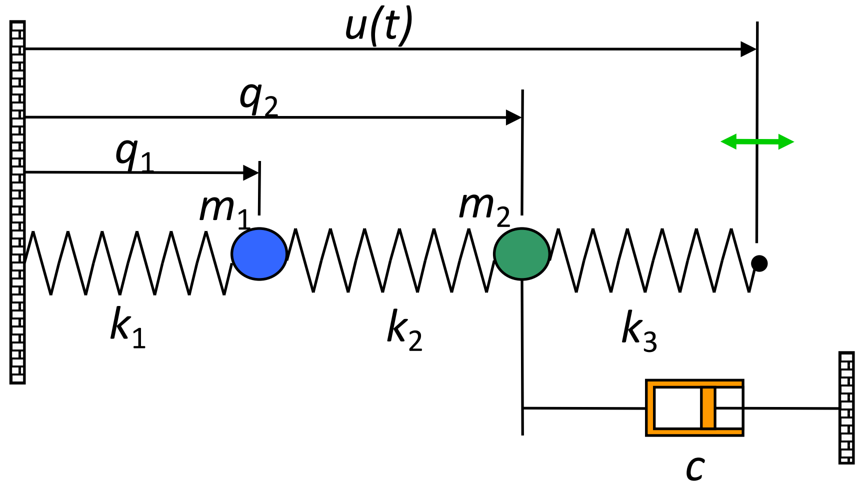


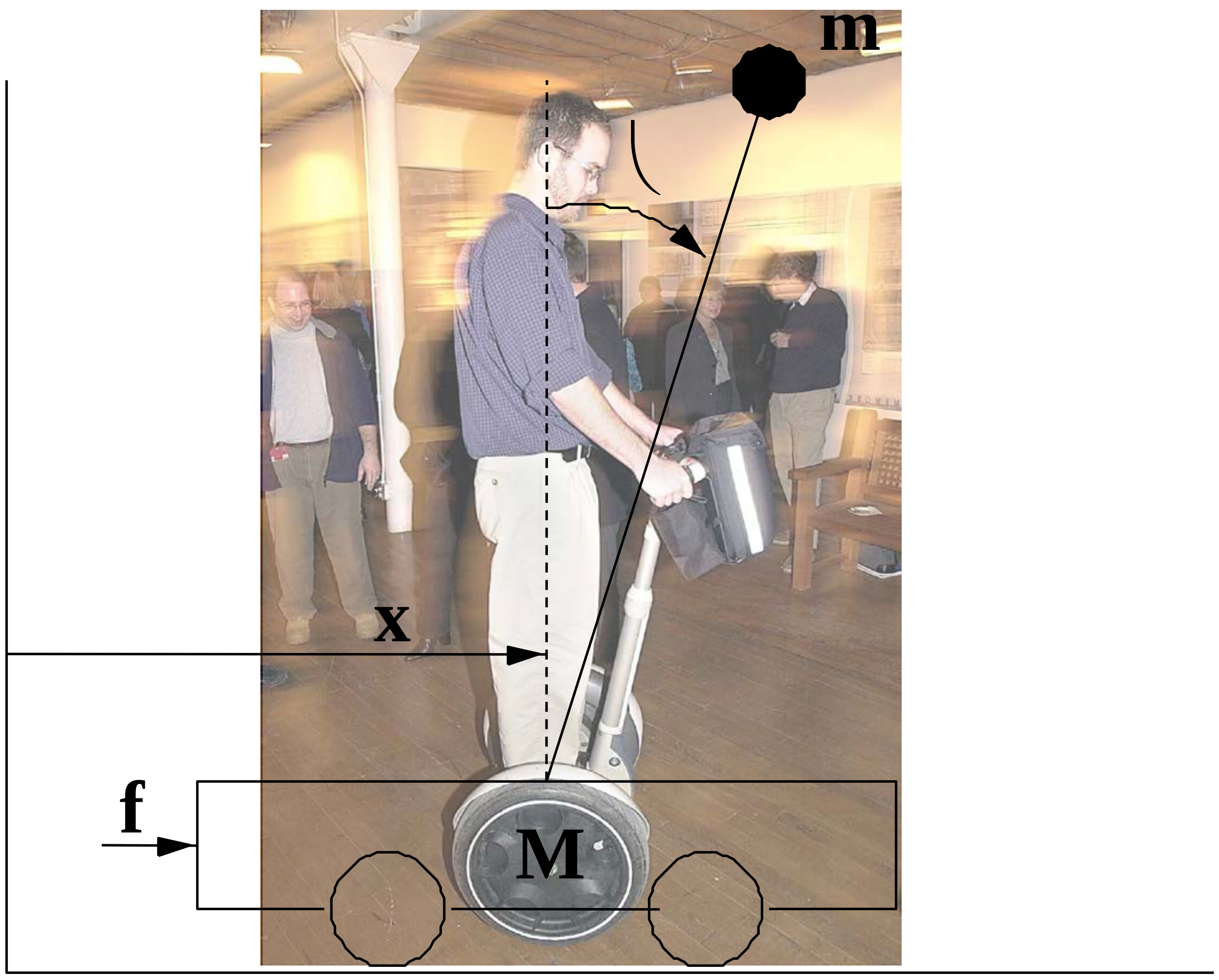
- Especially true after feedback
- Frequency response is key performance specification

Why are Linear Systems Important?

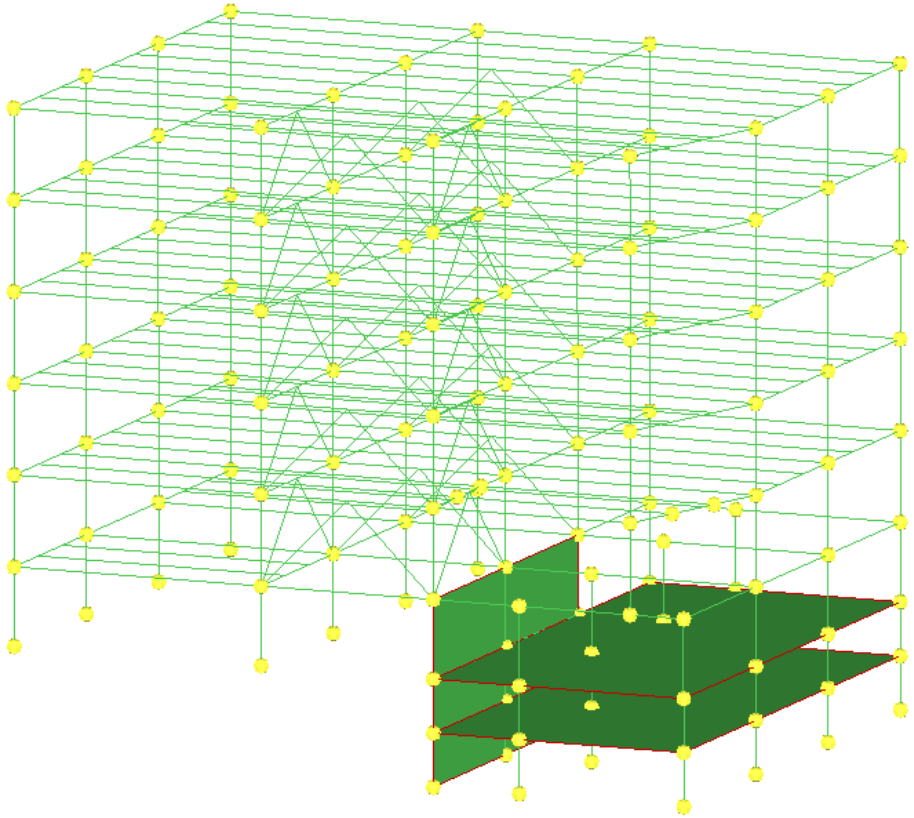
Many important *examples*

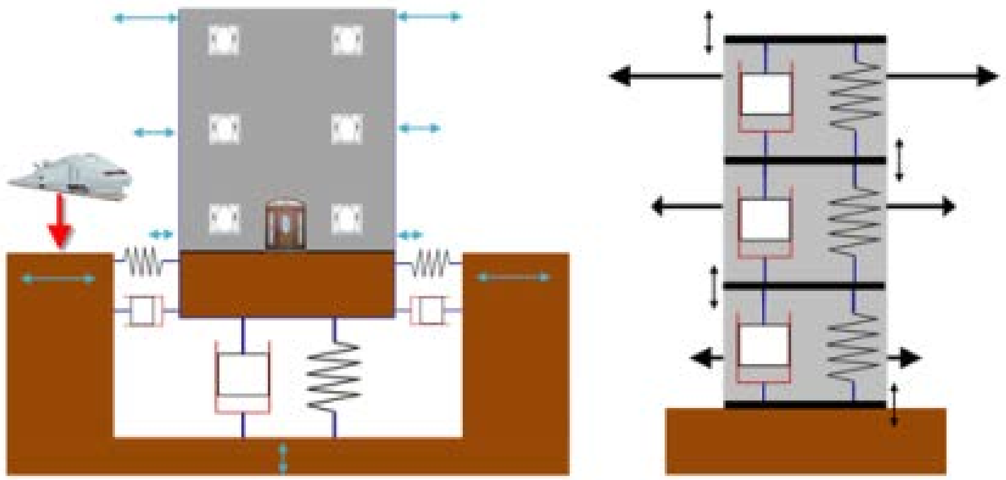
- Mechanical Systems

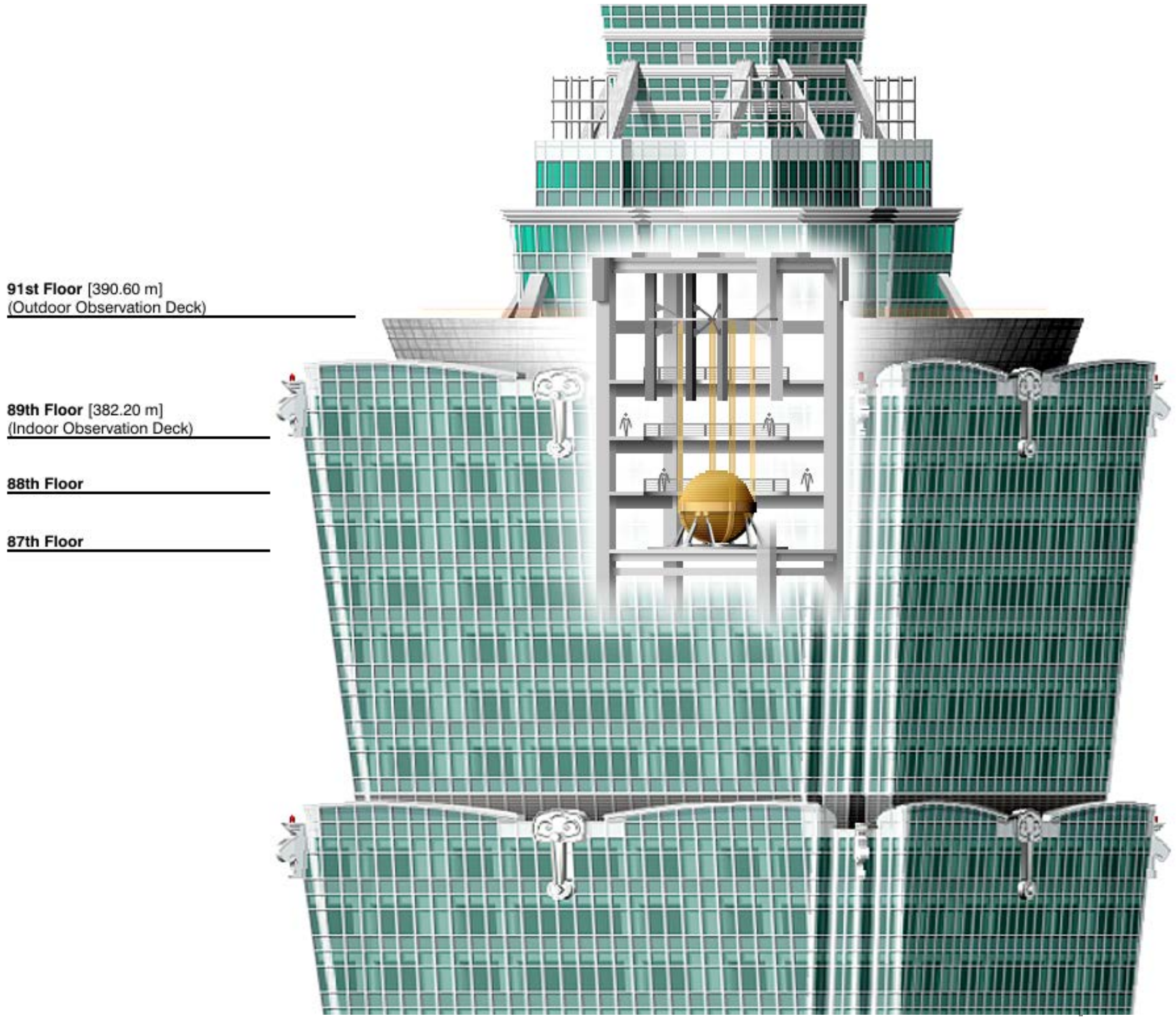




Why are Linear Systems Important?



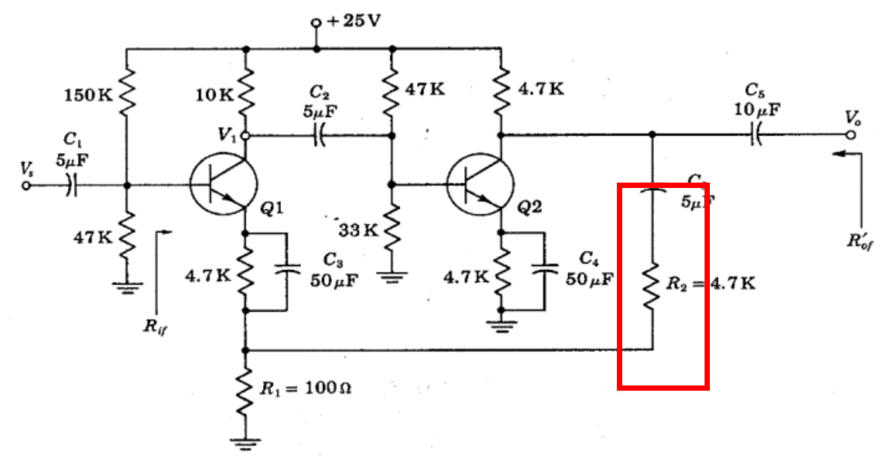


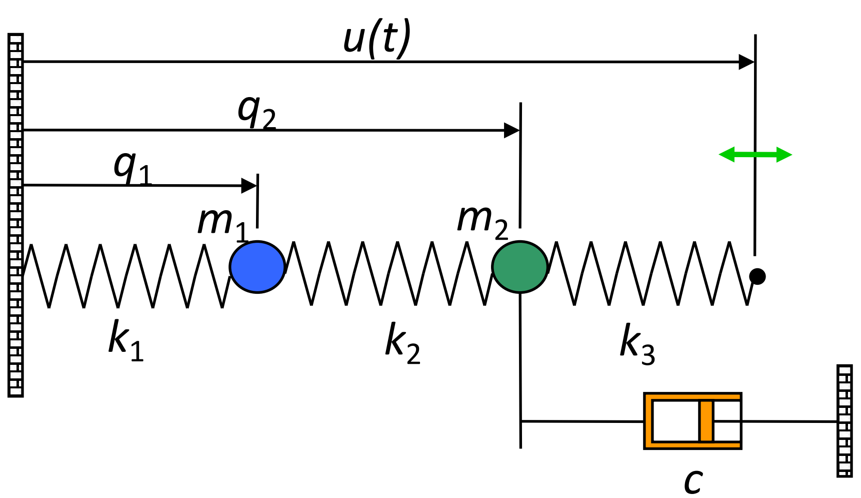


Why are Linear Systems Important?

Many important *examples*

- Electronic circuits





Many important *tools*

- Frequency and step response,
  - Traditional tools of control theory
  - Developed in 1930's at Bell Labs
- Classical control design toolbox
  - Nyquist plots, gain/phase margin
  - Loop shaping

CDS 110

- Optimal control and estimators
  - Linear quadratic regulators
  - Kalman estimators

CDS 112

- Robust control design
  - H_1 control design
  - μ analysis for structured uncertainty

CDS 212/ 213

Solutions of Linear Time Invariant Systems: The Matrix Exponential

Linear Time Invariant (LTI) System:

- If *Linear System* input u(t) leads to output y(t)
- If u(t+T) leads to output y(t+T), the system is *time invariant*

Scalar LTI system, with no input

$$\dot{x} = ax \qquad y = cx \qquad x(0) = x_0 \longrightarrow x(t) = e^{at}x_0 \longrightarrow y(t) = ce^{at}x_0$$

- Matrix LTI system, with no input

$$\dot{x} = Ax \qquad y = Cx \qquad x(0) = x_0 \longrightarrow x(t) = e^{At}x_0 \longrightarrow \boxed{y(t) = Ce^{At}x_0}$$

Matrix Exponential

The Matrix Exponential

Recall scalar exponential formula:

- $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots, \ -\infty < x < \infty$

Matrix exponential defined analogously:

- $e^M = I + \frac{1}{1!}M + \frac{1}{2!}M^2 + \frac{1}{3!}M^3 + \cdots = \sum_{k=0}^{\infty} \frac{1}{k!}M^k$

Some useful properties of the Matrix Exponential

- If $M = \begin{bmatrix} m_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & m_{nn} \end{bmatrix}$, then $e^M = \begin{bmatrix} e^{m_{11}} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & e^{m_{nn}} \end{bmatrix}$
- If matrix Y is invertible, then $e^{YMY^{-1}} = Ye^MY^{-1}$
- If M is diagonalizable ($M = TDT^{-1}$), then $e^M = Te^DT^{-1}$

The Convolution Integral: Step 1

Let $H(t)$ denote the response of a LTI system to a ***unit step*** input at t=0.

- Assuming the system starts at Equilibrium

The response to the steps are:

- First step input at time t=0: $H(t-t_0)u(t_0)$
- Second step input at time t_1: $H(t-t_1)(u(t_1)-u(t_0))$
- Third step input at time t_2: $H(t-t_2)(u(t_2)-u(t_1))$

By linearity, we can add the response

$$y(t) = H(t-t_0)u(t_0) + H(t-t_1)\big(u(t_1)-u(t_0)\big) + \ \ldots$$

$$= \big(H(t-t_0) - H(t-t_1)\big)u(t_0) + \big(H(t-t_1) - H(t-t_2)\big)u(t_1) + \ldots$$

$$= \sum_{n=0}^{t_0<t} \frac{H(t-t_n) - H(t-t_{n+1})}{t_{n+1}-t_n} u(t_n)(t_{n+1}-t_n)$$

Taking the limit as $(t_{n+1}-t_n) \rightarrow 0$

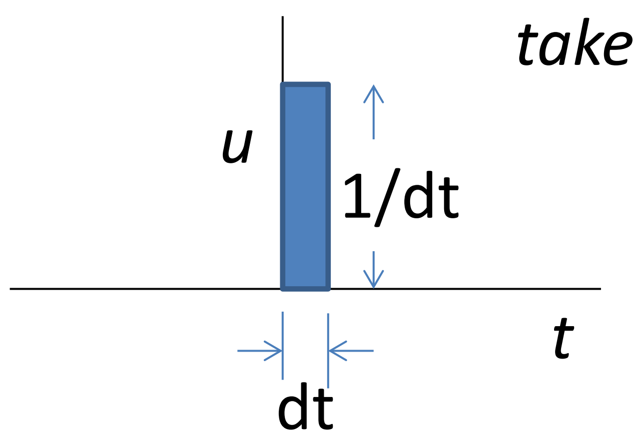
$$y(t) = \int_0^t H'\,(t-\tau)u(\tau)d\tau$$

Impulse Response

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \longrightarrow \quad y(t) = \underbrace{Ce^{At}x(0)}_{\text{homogeneous}} + ???$$

- What is the “impulse response” due to u(t)=δ(t)?

take limit as dt! 0 but keep unit area



- Apply this unit impulse to the system (with x(0)=0):

$$x(0^+) = \int_{0^-}^{0^+} (Ax + Bu)\mathrm{dt} = B$$

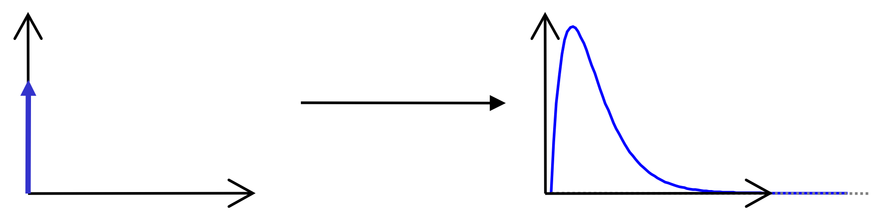
$$\Rightarrow \quad x(t) = e^{At}B$$

- Analogous to discrete-time response to input at time zero

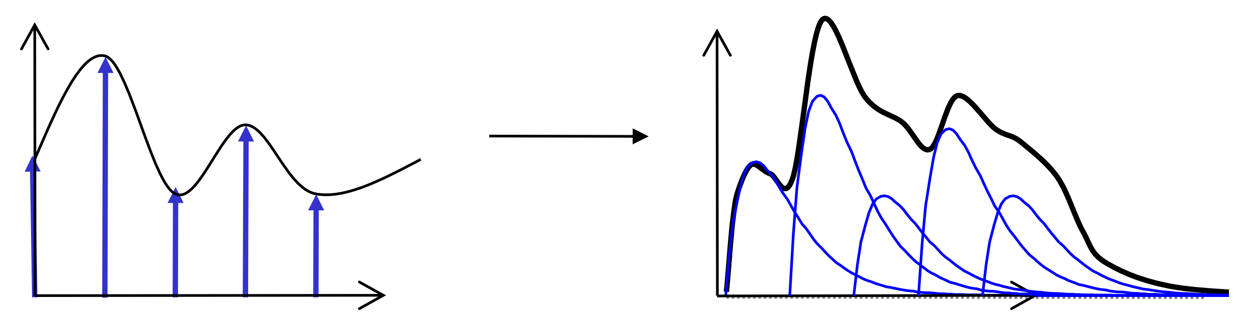
Response to inputs: Convolution

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \quad \longrightarrow \quad y(t) = \underbrace{Ce^{At}x(0)}_{\text{homogeneous}} + ???$$

- *Impulse response, $h(t) = Ce^{At}B$*
  - Response to input “impulse”
  - Equivalent to “Green’s function”



- Linearity $\Rightarrow$ compose response to arbitrary $u(t)$ using *convolution*
  - Decompose input into “sum” of shifted impulse functions
  - Compute impulse response for each
  - “Sum” impulse response to find $y(t)$
  - Take limit as dt$\rightarrow$ 0



- Complete solution: use integral instead of “sum”

$$y(t) = Ce^{At}x(0) + \int_{\tau=0}^{t} Ce^{A(t-\tau)}Bu(\tau)\mathrm{d}\tau + Du(t)$$

- linear with respect to initial condition *and* input
- 2X input $\Rightarrow$ 2X output when $x(0) = 0$

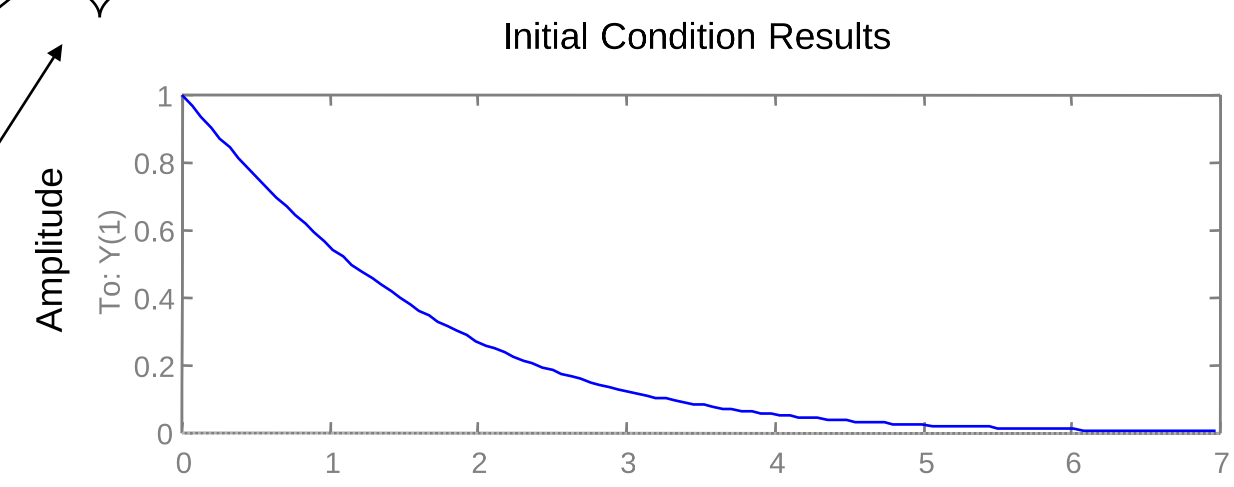
Matlab/Python Tools for Linear Systems

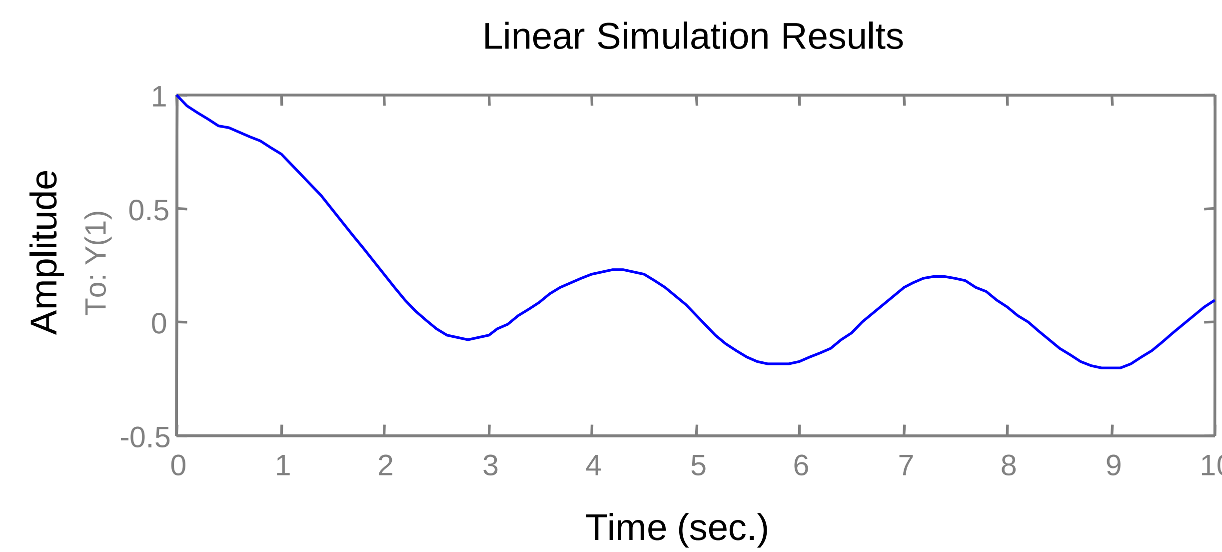
$$y(t) = \underbrace{\underbrace{Ce^{At}}x(0) + \int_{\tau=0}^{t} \underbrace{Ce^{A(t-\tau)}}Bu(\tau)\mathrm{d}\tau + Du(t)}$$

```
A = [-1 1; 0 -1];  B = [0; 1];
C = [1 0]; D = [0];
x0 = [1; 0.5];

sys = ss(A,B,C,D);
initial(sys, x0);
impulse(sys);

t = 0:0.1:10;
u = 0.2*sin(5*t) + cos(2*t);
lsim(sys, u, t, x0);
```





- Other MATLAB commands
  - gensig, square, sawtooth – produce signals of diff. types
  - step, impulse, initial, lsim – time domain analysis
  - bode, freqresp, evalfr – frequency domain analysis

ltiview – linear time invariant system plots

Input/Output Performance

Return to system with inputs

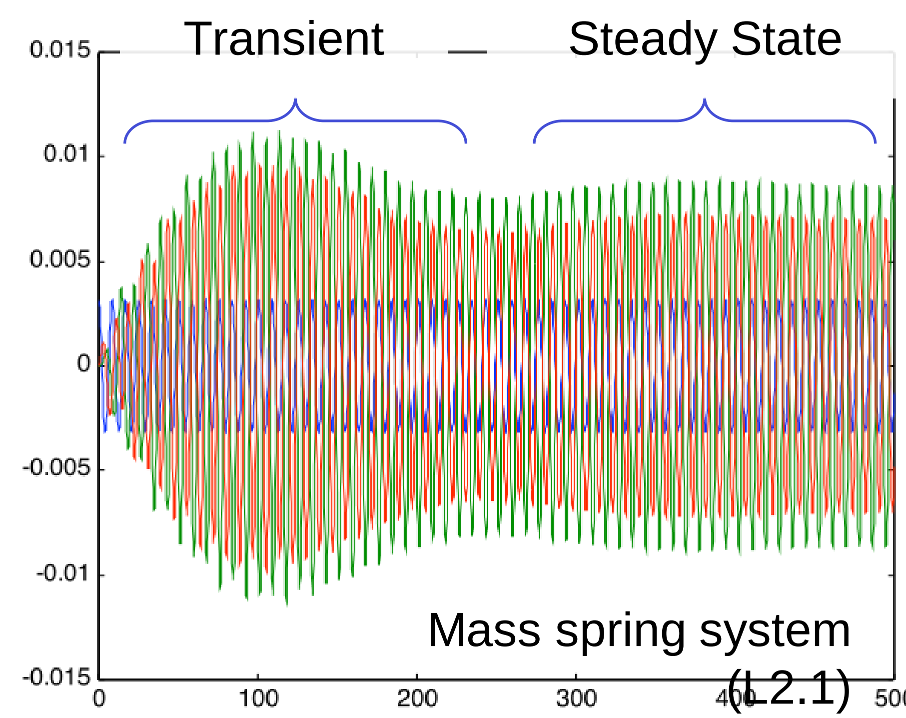
- How does system response to changes in input values?

Transient response:

- What happens right after a new input is applied

Steady state response:

- What happens a long time after the input is applied

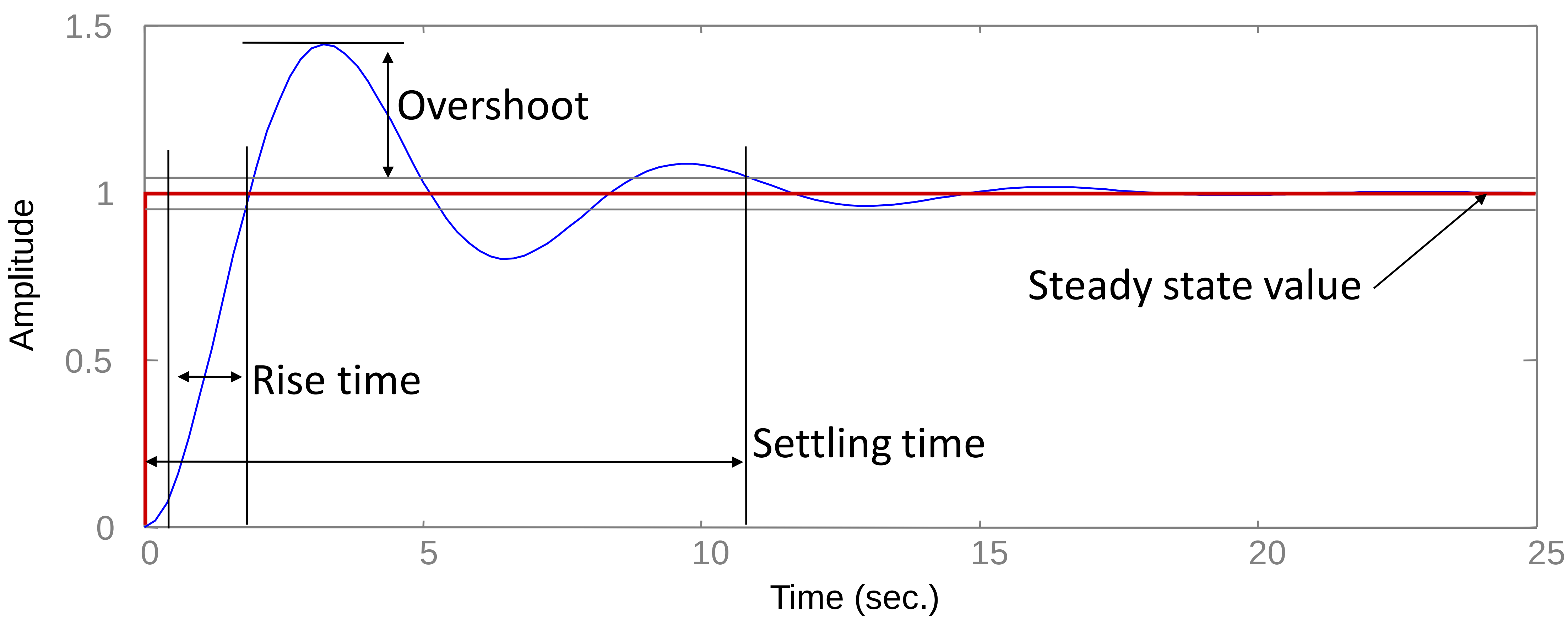


Stability vs input/output performance

- Systems that are close to instability typically exhibit poor input/output performan
- Nearly unstable systems (slow convergence) often exhibit “ringing” (highly oscillatory response to [non-periodic] inputs)

Step Response

- Output characteristics in response to a "step" input
  - Rise time: time required to move from 5% to 95% of final value
  - Overshoot: ratio between amplitude of first peak and steady state value
  - Settling time: time required to remain w/in *p*% (usually 2%) of final value
  - Steady state value: final value at t = 1



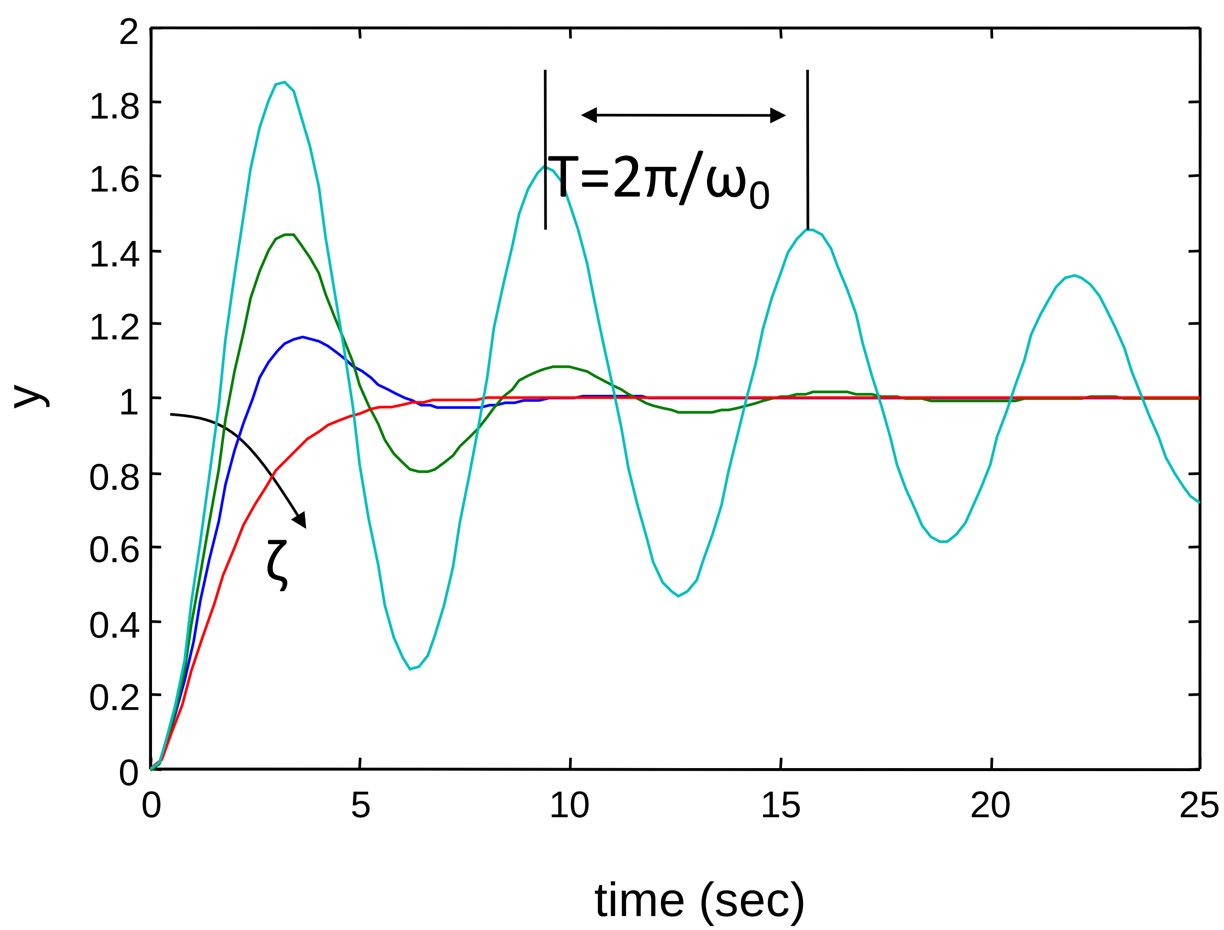
Second Order Systems

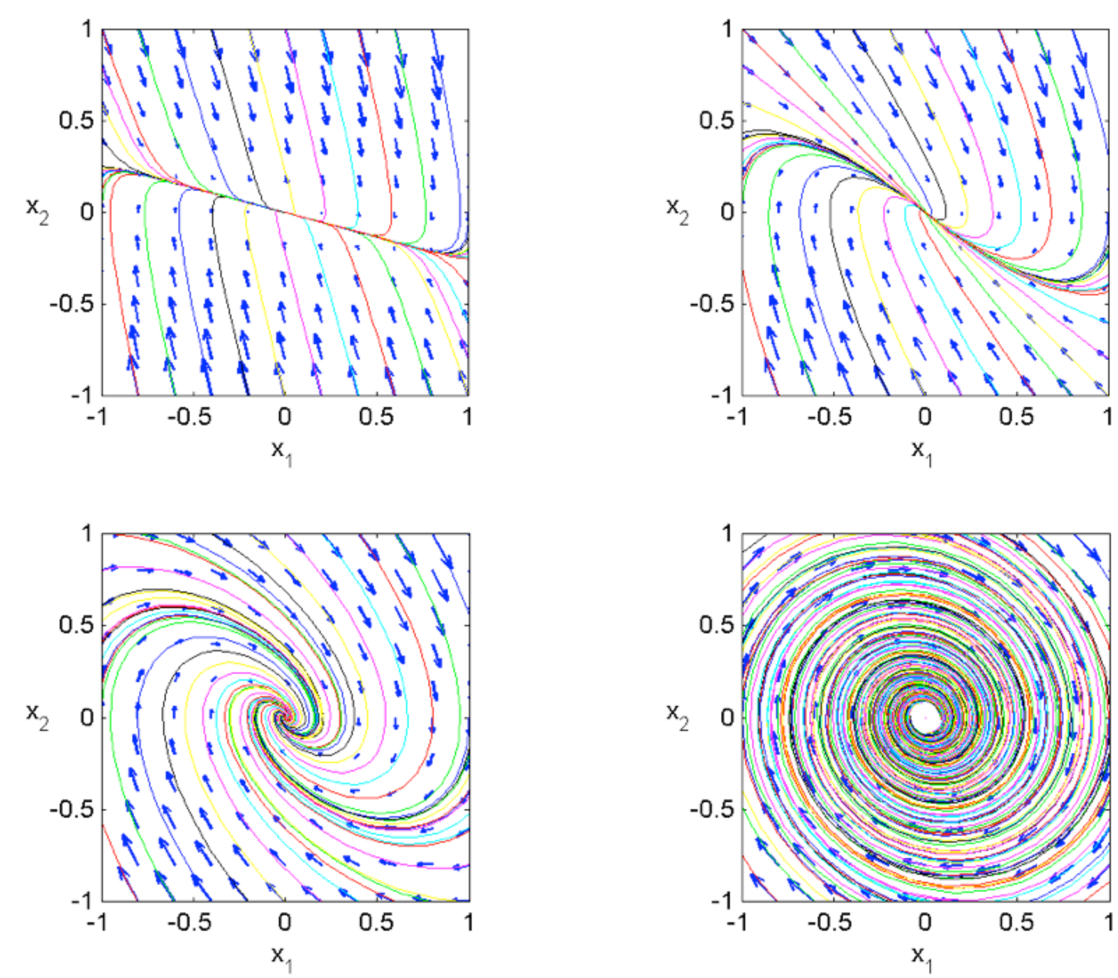
- Many important examples:
- Insight to response for higher orders (eigenvalues of A are either real or complex)
  - Exception is non-diagonalizable A (non-trivial Jordan form)

$$\ddot{q} + 2\zeta\omega_0\dot{q} + \omega_0^2 q = u \quad \leftrightarrow \quad \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & -2\zeta\omega_0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

For $\zeta < 1$, eigenvalues at

$$\left(-\zeta \pm j\sqrt{1-\zeta^2}\right)\omega_0$$





- Analytical formulas exist for overshoot, rise time, settling time, etc
- Will study more next week

Stability of Linear Systems

$$\dot{x} = Ax + Bu \quad (Bu \to 0)$$

$$\sout{y = Cx + Du}$$

$$x(t) = e^{At} x_0$$

Q: when is the system asymptotically stable?

$$\lim_{t \to \infty} x(t) = 0$$

Stability is determined by the *eigenvalues* of the matrix *A*

- Simple case: diagonal system

$$\dot{x} = \begin{bmatrix} \lambda_1 & & 0 \\ & \text{O} & \\ 0 & & \lambda_n \end{bmatrix} x \quad ® \quad x(t) = \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & \text{O} & \\ 0 & & e^{\lambda_n t} \end{bmatrix} x_0$$

Stable if $\lambda_i \leq 0$
Asy stable if $\lambda_i < 0$
Unstable if $\lambda_i > 0$

- More generally: transform to “Jordan” form

$$\dot{x} = T^{-1} J T x \qquad J = \begin{bmatrix} J_1 & & 0 \\ & \text{O} & \\ 0 & & J_k \end{bmatrix} \qquad J_i = \begin{bmatrix} \lambda_i & 1 & 0 \\ & \text{O} & 1 \\ 0 & & \lambda_i \end{bmatrix}$$

Asy stable if $\text{Re}(\lambda_i) < 0$
Unstable if $\text{Re}(\lambda_i) > 0$
Indeterminate if $\text{Re}(\lambda_i) = 0$

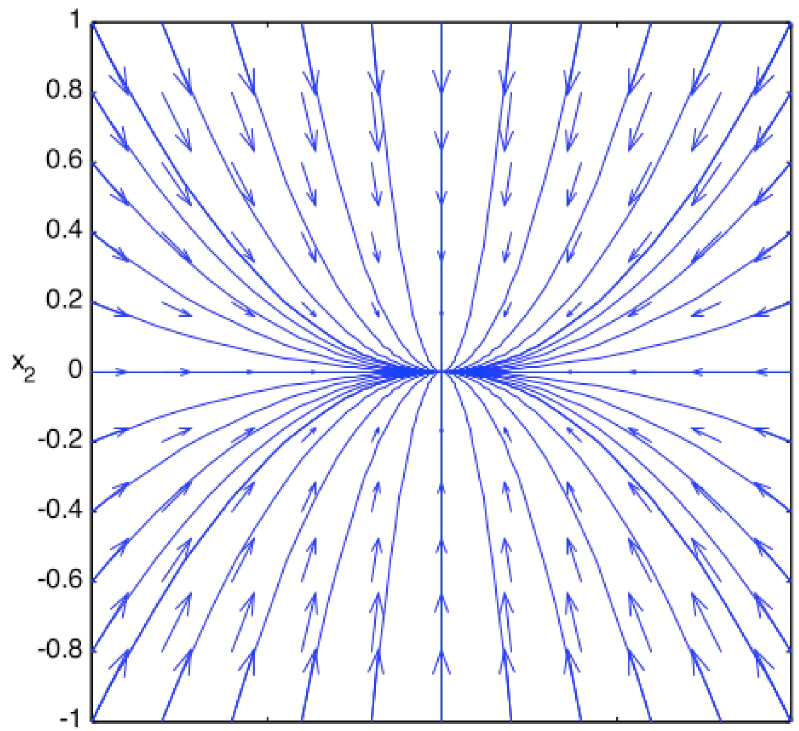
Form of eigenvalues determines system behavior

Linear systems are automatically *globally* stable or unstable

Eigenstructure of Linear Systems

Real e-values

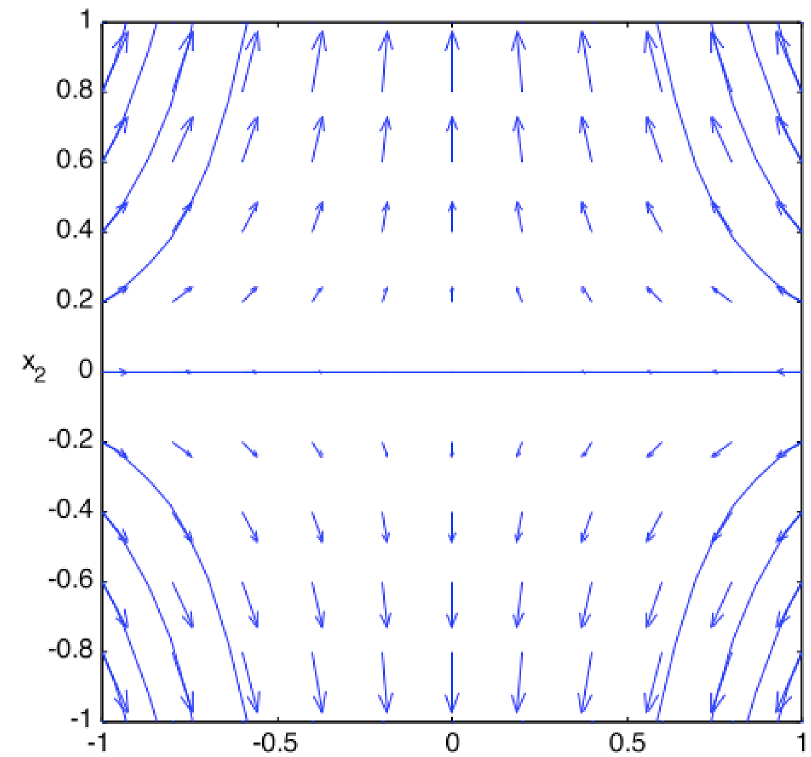
$\mathrm{Re}(\lambda_i) < 0$



Real e-values

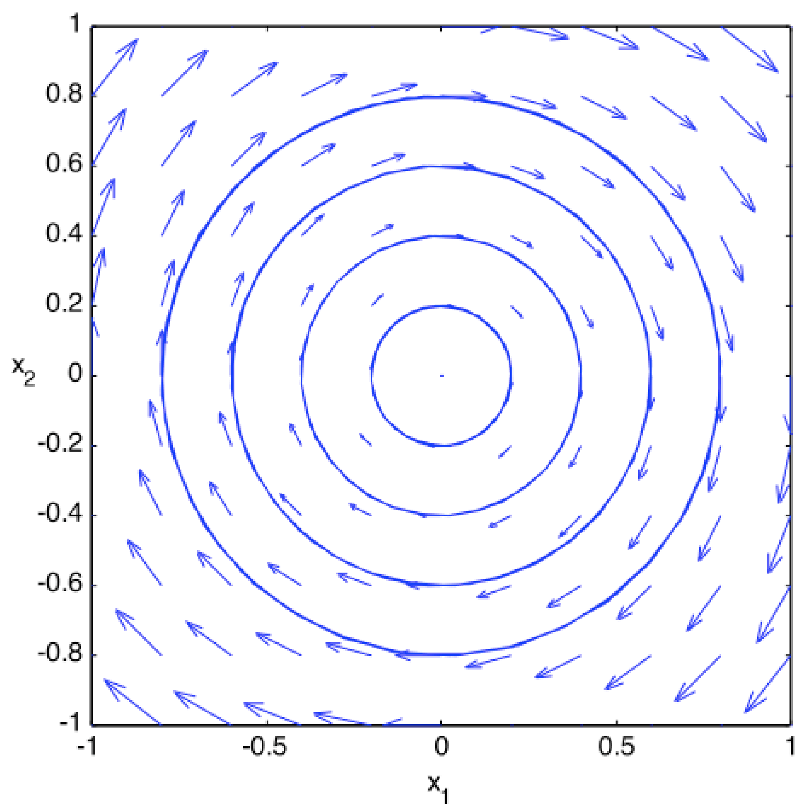
$\mathrm{Re}(\lambda_i) < 0$

$\mathrm{Re}(\lambda_i) > 0$



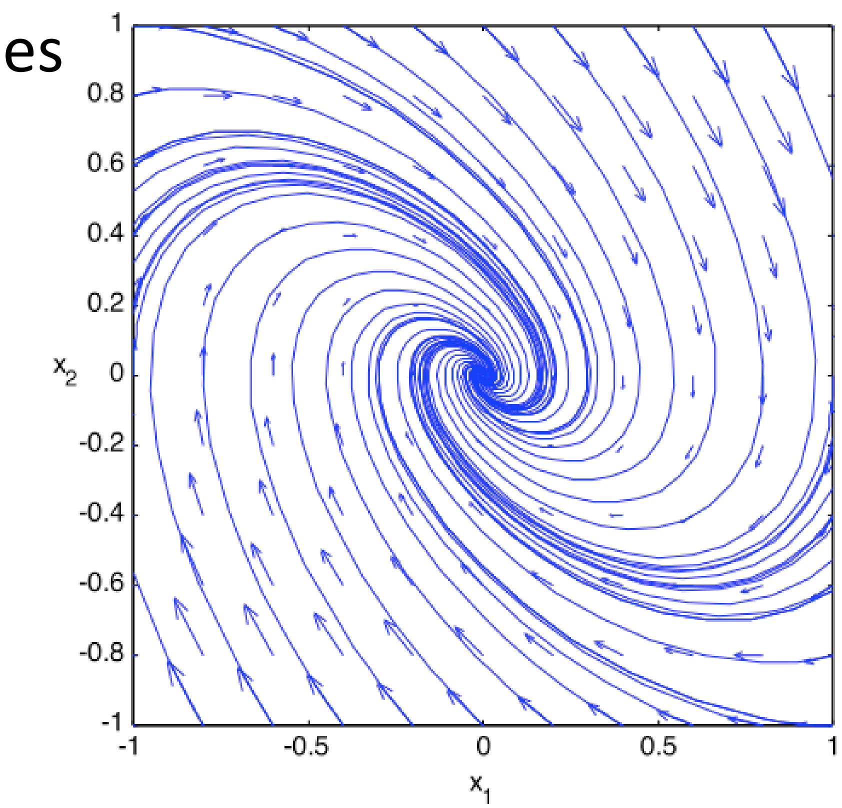
Complex e-values

$\mathrm{Re}(\lambda_i) = 0$



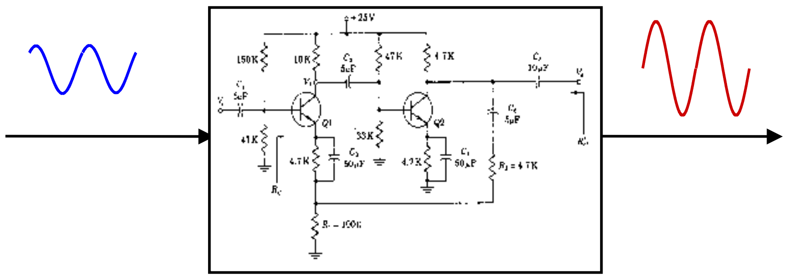
Complex e-values

$\mathrm{Re}(\lambda_i) < 0$

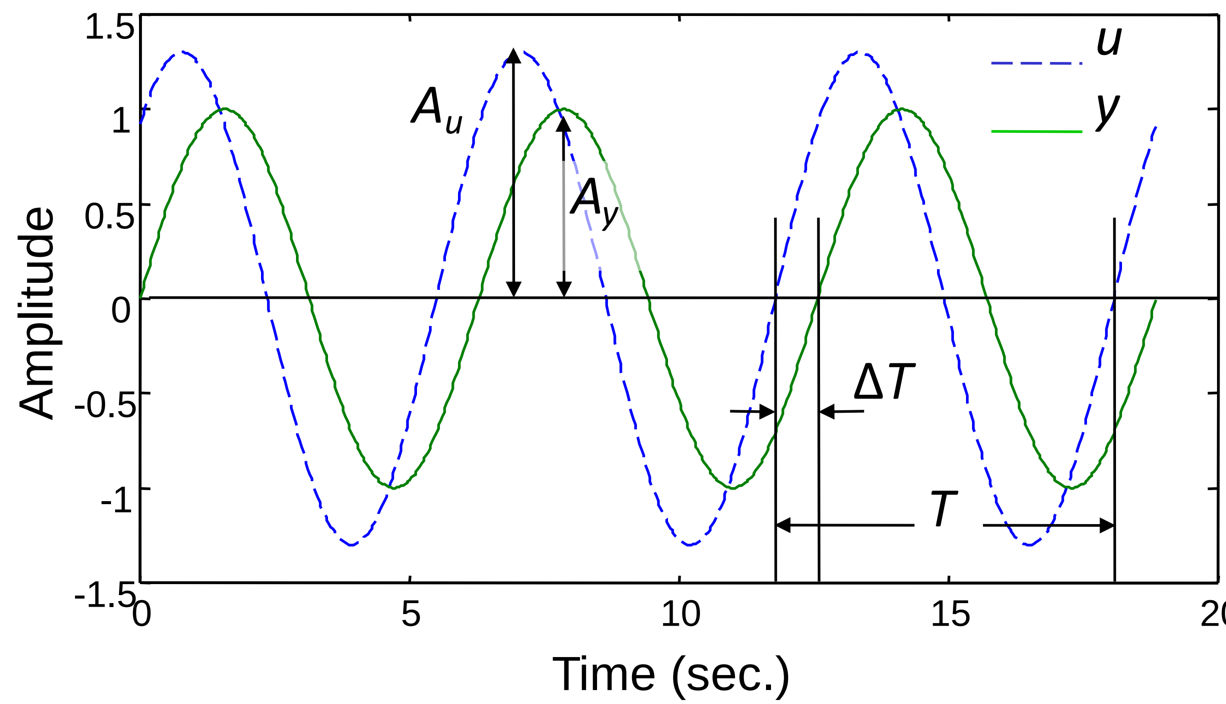


Frequency Response

- Measure *steady state* response of system to sinusoidal input
  - Example: audio amplifier – would like consistent ("flat") amplification between 20 Hz & 20,000 Hz
  - Individual sinusoids are good *test signals* for measuring performance in many systems



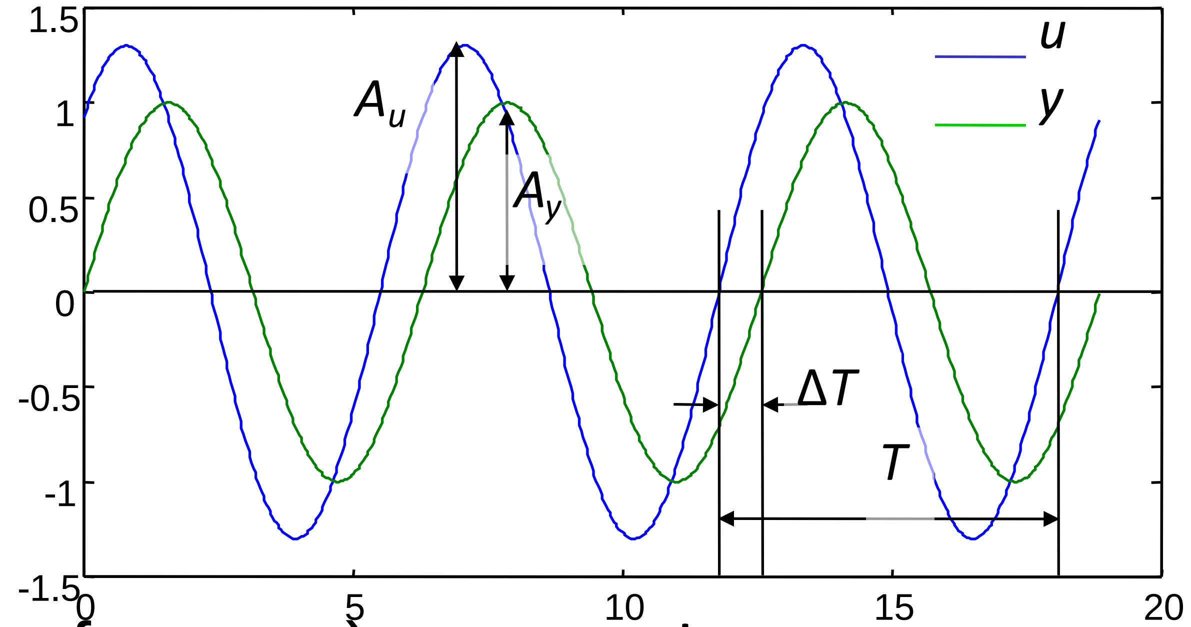
- Approach: plot input and output, measure *relative* amplitude and phase
  - Use MATLAB or SIMULINK to generate response of system to sinusoidal output
  - Gain = A_y/A_u
  - Phase = $2\pi \cdot \Delta T/T$



- May not work for *nonlinear* systems
  - System nonlinearities can cause *harmonics* to appear in the output
  - Amplitude and phase may not be well-defined
  - For *linear* systems, frequency response is always well defined

Computing Frequency Responses

- Technique #1: plot input and output, measure relative amplitude and phase
  - Generate response of system to sinusoidal output
  - Gain = A_y/A_u
  - Phase = $2\pi \cdot \Delta T/T$
  - For *linear* system, gain and phase don't depend on the input amplitude

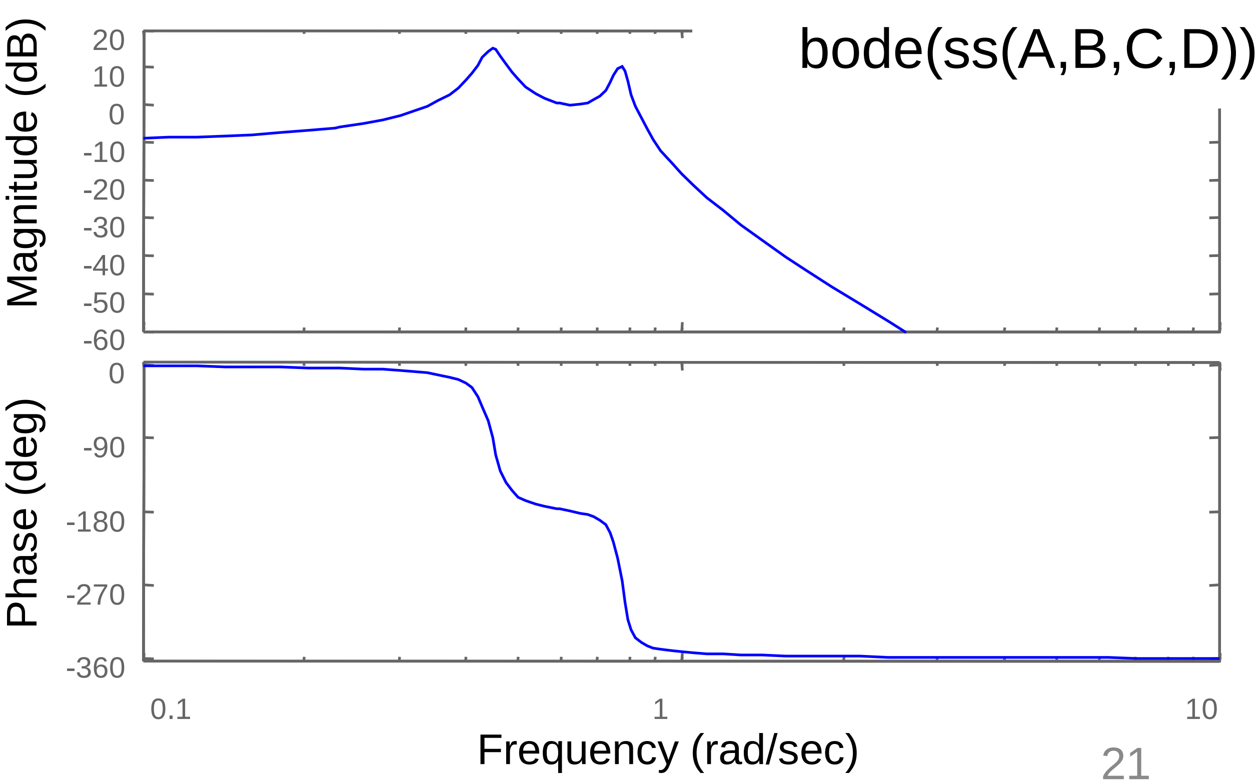


- Technique #2 (linear systems): use bode (or freqresp) command
  - Assumes linear dynamics in state space form:

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

  - Gain plotted on log-log scale
    - dB = 20 $\log_{10}$ (gain)
  - Phase plotted on linear-log scale



Calculating Frequency Response from convolution equation… (more later)

- Convolution equation describes response to any input; use this to look at response to sinusoidal input: $u(t) = A\sin(\omega t) = \frac{A}{2i}\left(e^{i\omega t} - e^{-i\omega t}\right)$

u(t)

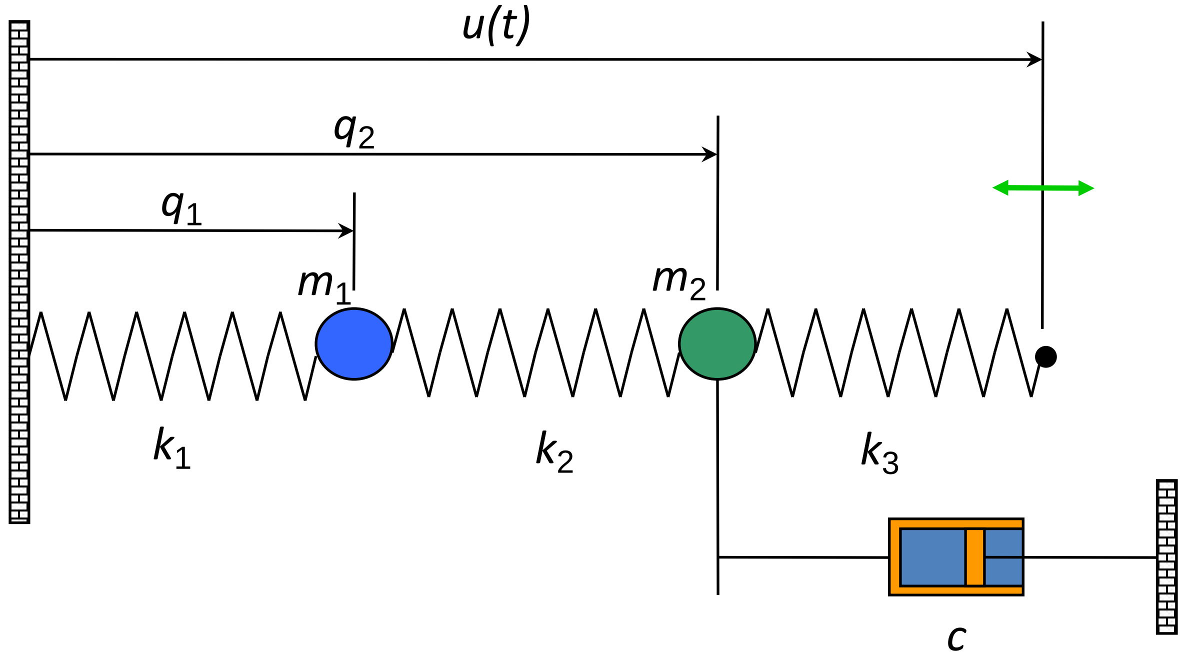
$$
\begin{aligned}
x(t) &= e^{At}x(0) + \int_0^t e^{A(t-\tau)}B\boxed{e^{i\omega\tau}}\mathrm{d}\tau \\
&= e^{At}x(0) + e^{At}\int_0^t e^{(i\omega I - A)\tau}B\mathrm{d}\tau \\
&= e^{At}x(0) + e^{At}(i\omega I - A)^{-1}e^{(i\omega I - A)\tau}\Big|_{\tau=0}^{t}B \\
&= e^{At}x(0) + e^{At}(i\omega I - A)^{-1}\left(e^{(i\omega I - A)t} - I\right)B \\
&= \underbrace{e^{At}\left(x(0) - (i\omega I - A)^{-1}B\right)}_{\text{Transient (decays if stable)}} + \underbrace{(i\omega I - A)^{-1}B}_{\text{Ratio of response/input}}e^{i\omega t}
\end{aligned}
$$

$$
\begin{aligned}
y(t) &= Cx(t) + Du(t) \\
&= Ce^{At}\left(x(0) - (i\omega I - A)^{-1}B\right) + \boxed{\left(C(i\omega I - A)^{-1}B + D\right)}e^{i\omega t}
\end{aligned}
$$

"Frequency response"

Spring Mass System

Frequency response: $C(j\omega I-A)^{-1}B+D$



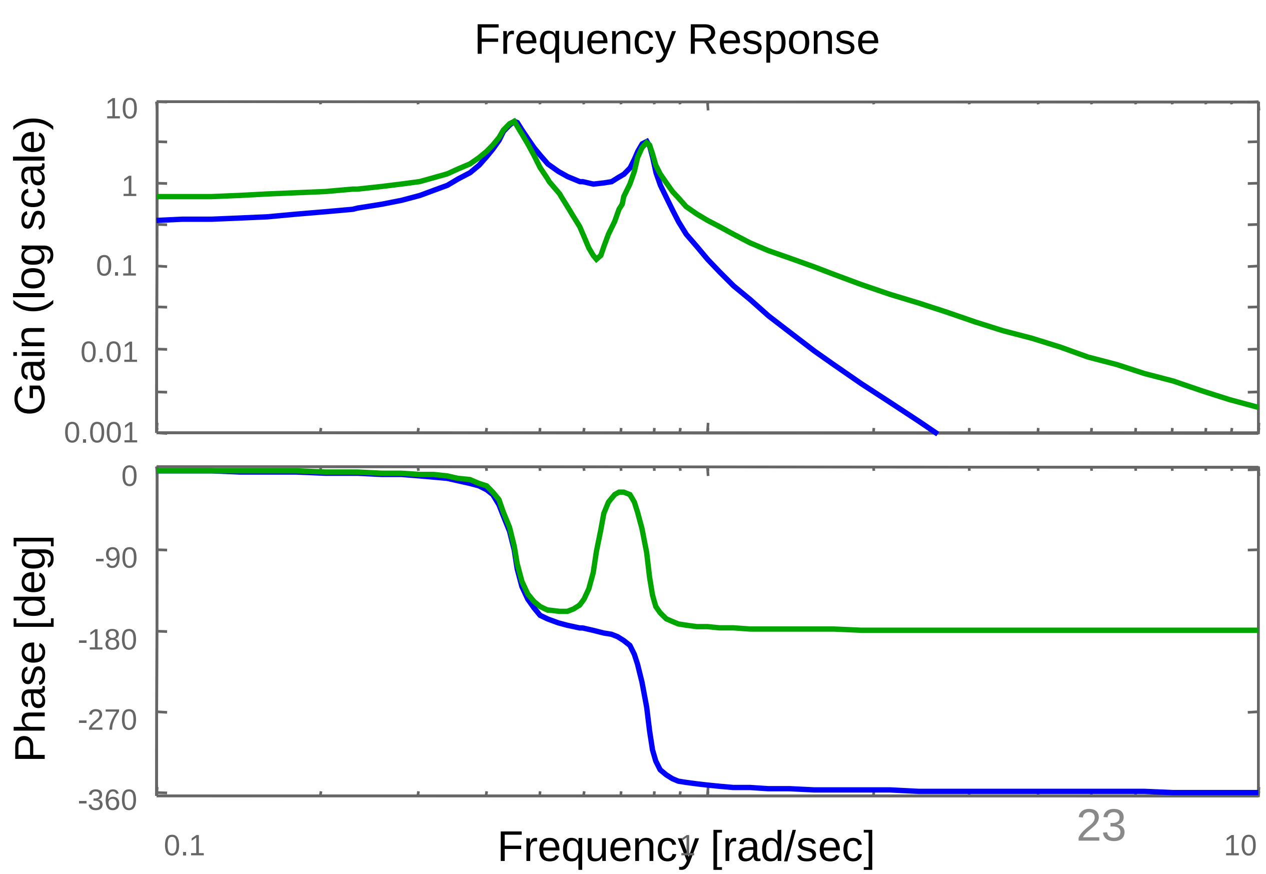
$$\frac{d}{dt}\begin{bmatrix} q_1 \\ q_2 \\ \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_1+k_2}{m} & \frac{k_2}{m} & 0 & 0 \\ \frac{k_2}{m} & -\frac{k_2+k_3}{m} & 0 & -\frac{c}{m} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}$$

With $k_1 = k_2 = 1,\ m = 1,\ c = 0$

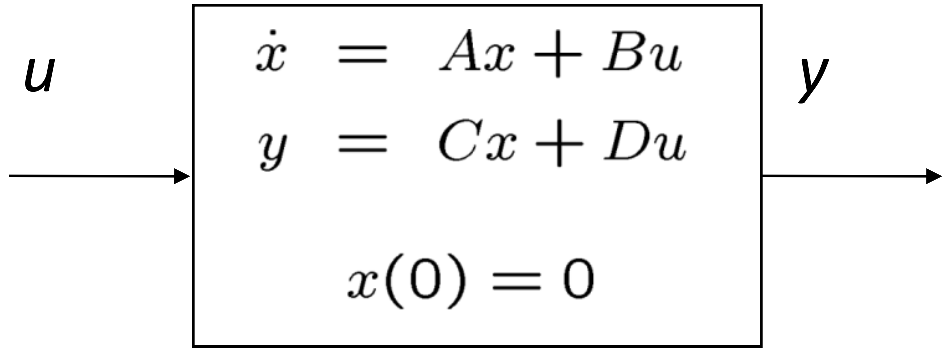
$$v_{1,2} = \begin{bmatrix} 1 \\ 1 \\ \pm 1i \\ \pm 1i \end{bmatrix} \qquad v_{3,4} = \begin{bmatrix} 1 \\ -1 \\ \pm\sqrt{2}i \\ \mp\sqrt{2}i \end{bmatrix}$$

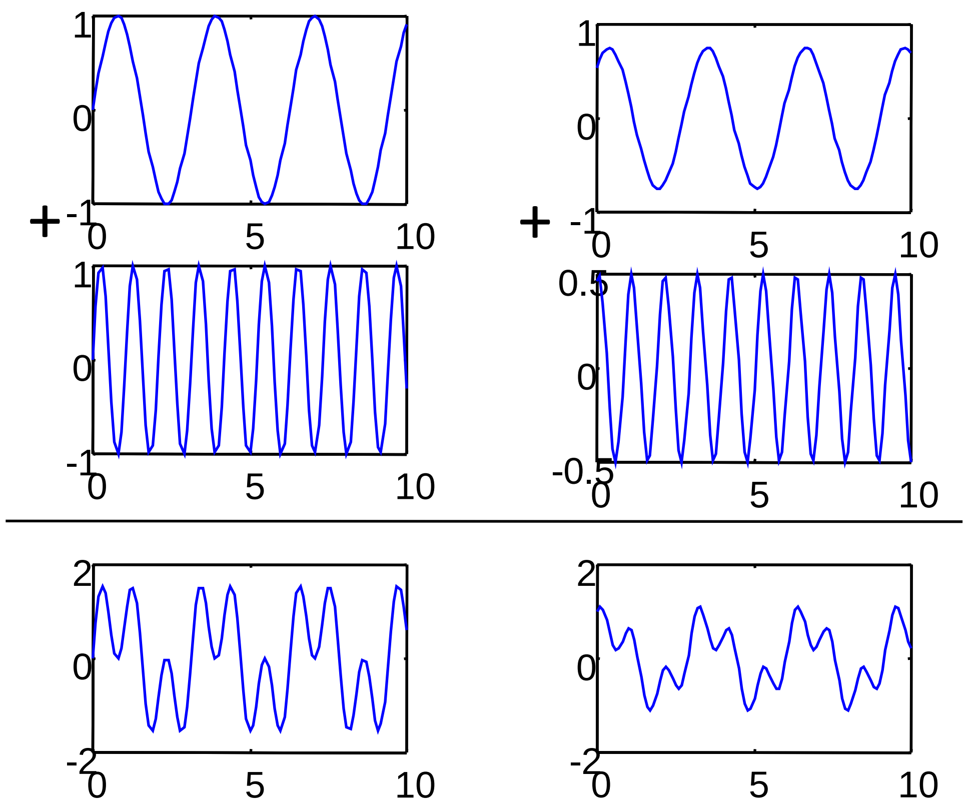
Eigenvalues of A:

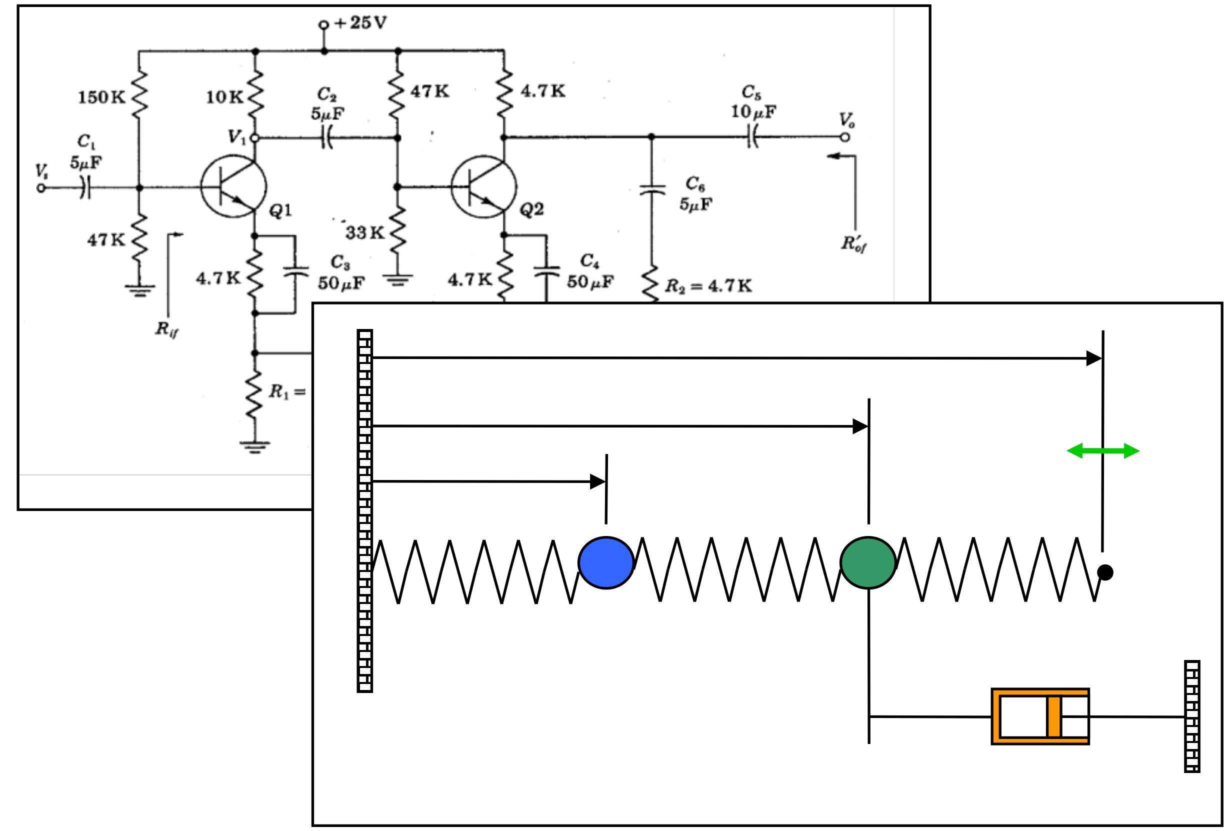
- For zero damping, $j\omega_1$ and $j\omega_2$
- ω_1 and ω_2 correspond frequency response peaks
- The eigenvectors for these eigenvalues give the *mode shape*:
  - In-phase motion for lower freq.
  - Out-of phase motion for higher freq.



Summary: Linear Systems







$$y(t) = Ce^{At}x(0) + \int_{\tau=0}^{t} Ce^{A(t-\tau)}Bu(\tau)\mathrm{d}\tau + Du(t)$$

- Properties of linear systems
  - Linearity with respect to initial condition and inputs
  - Stability characterized by eigenvalues
  - Many applications and tools available
  - Provide local description for nonlinear systems

Linearization Around an Equilibrium Point

$$\dot{x} = f(x,u) \qquad \dot{z} = Az + Bv$$
$$y = h(x,u) \quad \longrightarrow \quad w = Cz + Dv$$

"Linearize" around $x = x_e$

$$f(x_e,u_e) = 0 \qquad y_e = h(x_e,u_e)$$

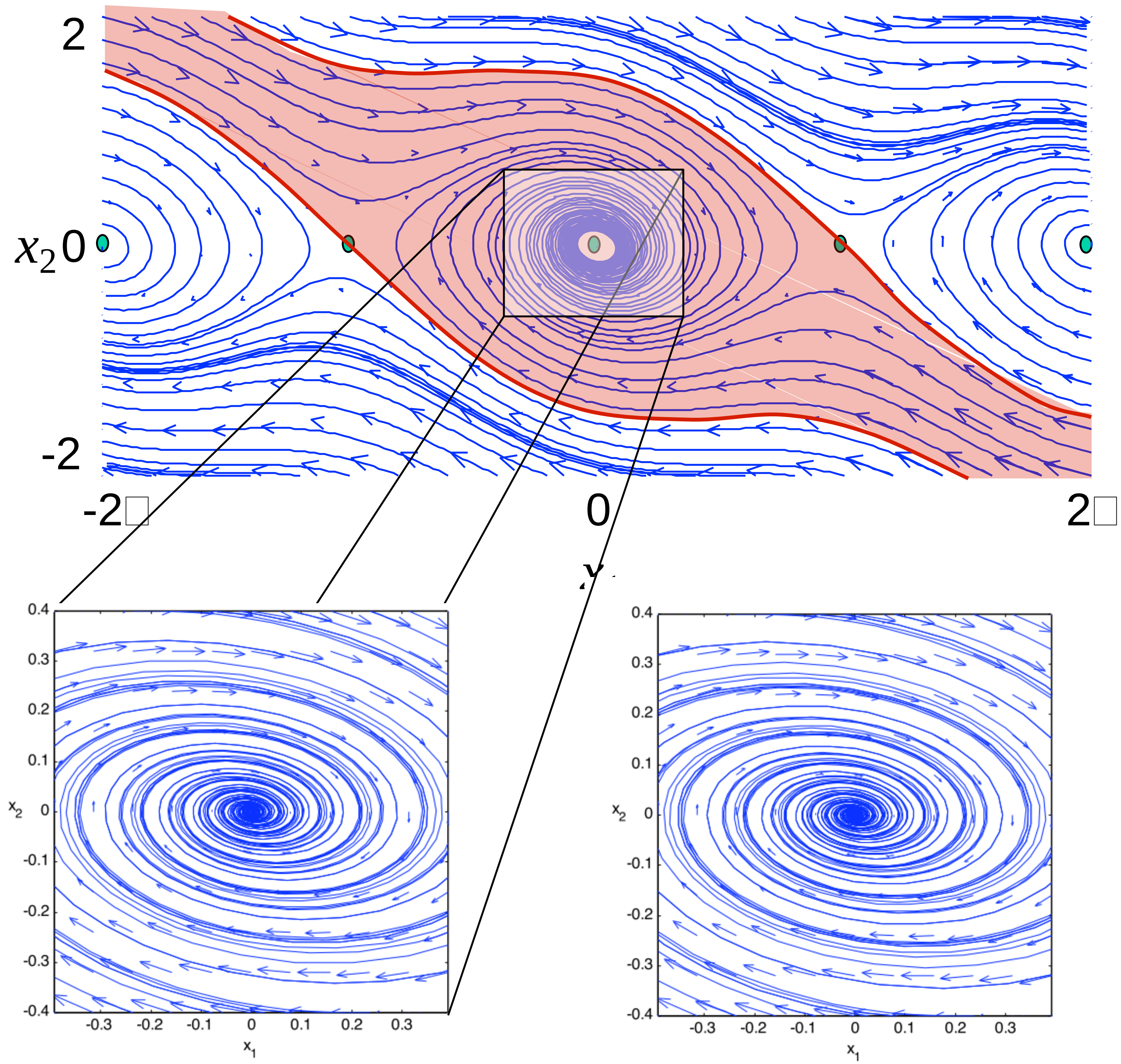
$$z = x - x_e \qquad v = u - u_e \qquad w = y - y_e$$

$$A = \left.\frac{\partial f}{\partial x}\right|_{(x_e,u_e)} \qquad B = \left.\frac{\partial f}{\partial u}\right|_{(x_e,u_e)}$$

$$C = \left.\frac{\partial h}{\partial x}\right|_{(x_e,u_e)} \qquad D = \left.\frac{\partial h}{\partial u}\right|_{(x_e,u_e)}$$

Remarks

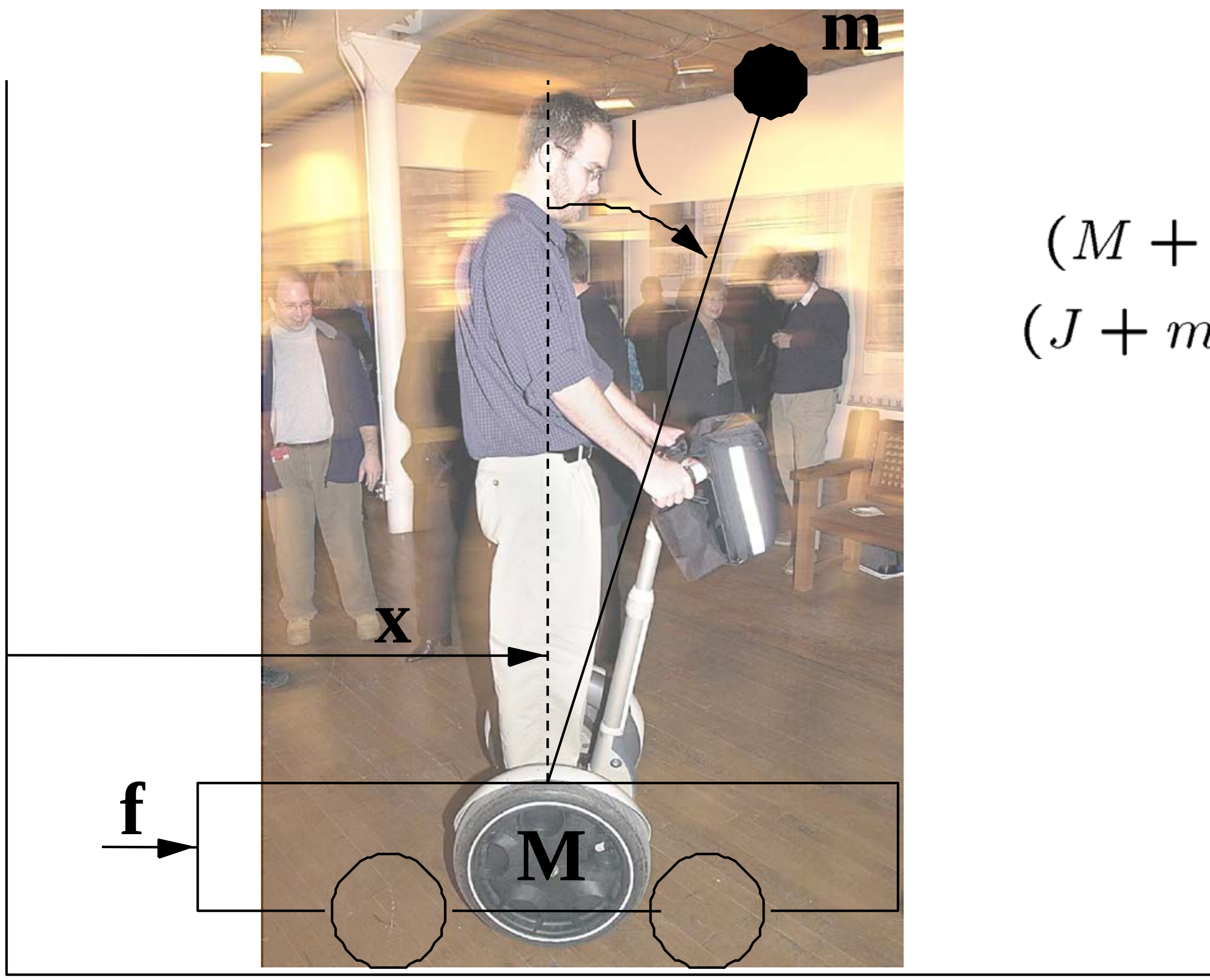
- In examples, this is often equivalent to small angle approximations, etc
- Only works *near* to equilibrium point



Full nonlinear model

Linear model (honest!)

Example: Inverted Pendulum on a Cart



$$(M+m)\ddot{x} + ml\cos\theta\ddot{\theta} = -b\dot{x} + ml\sin\theta\dot{\theta}^2 + f$$
$$(J+ml^2)\ddot{\theta} + ml\cos\theta\ddot{x} = -mgl\sin\theta$$

- State: $x, \theta, \dot{x}, \dot{\theta}$
- Input: $u = F$
- Output: $y = x$
- Linearize according to previous formula around $\theta = 0$

$$\frac{d}{dt}\begin{bmatrix} x \\ \theta \\ \dot{x} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{m^2gl^2}{J(M+m)+Mml^2} & \frac{-(J+ml^2)b}{J(M+m)+Mml^2} & 0 \\ 0 & \frac{mgl(M+m)}{J(M+m)+Mml^2} & \frac{-mlb}{J(M+m)+Mml^2} & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ \frac{J+ml^2}{J(M+m)+Mml^2} \\ \frac{ml}{J(M+m)+Mml^2} \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} x$$