CDS 101/110: Lecture 10-1
Limits on Performance

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Goals:
• Describe limits of performance on feedback systems
• Introduce Bode’s integral formula and the “waterbed” effect
• Show some of the limitations of feedback due to RHP poles and zeros

Reading:
• Åström and Murray, Feedback Systems, Section 12.6

Algebraic Constraints on Performance

Goal: keep $S$ & $T$ small
• $S$ small $\Rightarrow$ low tracking error
• $T$ small $\Rightarrow$ good noise rejection (and robustness [CDS 112/212])

Problem: $S + T = 1$
• Can’t make both $S$ & $T$ small at the same frequency
• Solution: keep $S$ small at low frequency and $T$ small at high frequency
• Loop gain interpretation: keep $L$ large at low frequency, and small at high frequency

Sensitivity function
$H_{er} = \frac{1}{1 + PC} = : S$

Complementary sensitivity function
$H_{yn} = \frac{PC}{1 + PC} = : T$

- Transition between large gain and small gain complicated by stability (phase margin)
Bode’s Integral Formula and the Waterbed Effect

Bode’s integral formula for \( S = 1/(1+PC) = 1/(1+L) \):
- Let \( p_k \) be the unstable poles of \( L(s) \) and assume relative degree of \( L(s) \) \( \geq 2 \)
- Theorem: the area under the sensitivity function is a conserved quantity:
\[
\int_0^\infty \log_e |S(j\omega)| \, d\omega = \int_0^\infty \log_e \left| \frac{1}{1 + L(j\omega)} \right| \, d\omega = \pi \sum \text{Re } p_k
\]

Waterbed effect:
- Making sensitivity smaller over some frequency range requires \textit{increase} in sensitivity someplace else
- Presence of RHP poles makes this effect worse
- Actuator bandwidth further limits what you can do
- Note: area formula is linear in \( \omega \); Bode plots are logarithmic

Example: Magnetic Levitation

System description
- Ball levitated by electromagnet
- Inputs: current thru electromagnet
- Outputs: position of ball (from IR sensor)
- States: \( z, \dot{z} \)
- Dynamics: \( F = ma, F = \text{magnetic force generated by wire coil} \)
- See MATLAB handout for details

Controller circuit
- Active R/C filter network
- Inputs: set point, disturbance, ball position
- States: currents and voltages
- Outputs: electromagnet current
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Equations of Motion

Process: actuation, sensing, dynamics
\[ m\ddot{z} = mg - k_m (k_A u)^2 / z^2 \]
\[ v_{ir} = k_T z + v_0 \]
- \( u \) = current to electromagnet
- \( v_{ir} \) = voltage from IR sensor

Linearization:
\[ P(s) = \frac{-k}{s^2 - r^2} \quad k, r > 0 \]
- Poles at \( s = \pm r \Rightarrow \) open loop unstable

Control Design

Need to create encirclement
- Loop shaping is not useful here
- Flip gain to bring Nyquist plot over -1 point
- Insert phase to create CCW encirclement

Can accomplish using a lead compensator
- Produce phase lead at crossover
- Generates loop in Nyquist plot
\[ C(s) = -k \frac{s + a}{s + b} \]

Note: RHP pole in L \( \Rightarrow \) need one net encirclement (CCW)
Performance Limits

Nominal design gives low perf
- Not enough gain at low frequency
- Try to adjust overall gain to improve low frequency response
- Works well at moderate gain, but notice waterbed effect

Bode integral limits improvement
\[ \int_0^\infty \log|S(j\omega)|d\omega = \pi r \]
- Must increase sensitivity at some point

Right Half Plane Zeros

Right half plane zeros produce “non-minimum phase” behavior
- Phase of frequency response has additional phase lag for given magnitude
- Can cause output to move opposite from input for a short period of time

Example: \[ H_1(s) = \frac{s + a}{s^2 + 2\zeta \omega_n s + \omega_n^2} \] vs \[ H_2(s) = \frac{s - a}{s^2 + 2\zeta \omega_n s + \omega_n^2} \]
Example: Lateral Control of the Ducted Fan

![Diagram of ducted fan](image)

\[ H_{\text{eff}}(s) = \frac{(s^2 - mgI)}{s^2(Js^2 + ds + mgI)} \]

- Poles: 0, 0, -α ± jωd
- Zeros: ±\(\sqrt{mgI}\)

Source of non-minimum phase behavior
- To move left, need to make \(\theta > 0\)
- To generate positive \(\theta\), need \(f_1 > 0\)
- Positive \(f_1\) causes fan to move right initially
- Fan starts to move left after short time (as fan rotates)

Stability in the Presence of Zeros

Loop gain limitations
- Poles of closed loop = poles of 1 + L. Suppose \(C = k nc/dc\), where \(k\) is the gain of the controller

\[ 1 + L = 1 + k \frac{ncn_p}{dcd_p} = \frac{dcd_p + kn_cnp}{dcd_p} \]

- For large \(k\), closed loop poles approach open loop zeros
- RHP zeros limit maximum gain ⇒ serious design constraint!

Root locus interpretation
- Plot location of eigenvalues as a function of the loop gain \(k\)
- Can show that closed loop poles go from open loop poles (\(k = 0\)) to open loop zeros (\(k = \text{infty}\))
Additional performance limits due to RHP zeros

Another waterbed-like effect: look at maximum of $H_{er}$ over frequency range:

$$M_1 = \max_{\omega_1 \leq \omega \leq \omega_2} |H_{er}(j\omega)| \quad M_2 = \max_{\theta} |H_{er}(j\omega)|$$

Thm: Suppose that $P$ has a RHP zero at $z$. Then there exist constants $c_1$ and $c_2$ (depending on $\omega_1, \omega_2, z$) such that $c_1 \log M_1 + c_2 \log M_2 \geq 0$

- $M_1$ typically $\ll 1 \Rightarrow M_2$ must be larger than 1 (since sum is positive)
- If we increase performance in active range (make $M_1$ and $H_{er}$ smaller), we must lose performance ($H_{er}$ increases) someplace else
- Note that this affects peaks not integrals (different from RHP poles)

$$H(s) = \frac{(s^2 - mgl)}{s^2(Js^2 + ds + mgl)}$$

- Poles: 0, 0, $-\sigma \pm j\omega_d$
- Zeros: $\pm \sqrt{mgl}$

Summary: Limits of Performance

Many limits to performance

- Algebraic: $S + T = 1$
- RHP poles: Bode integral formula
- RHP zeros: Waterbed effect on peak of $S$

Main message: try to avoid RHP poles and zeros whenever possible (e.g., re-design)

$$\int_0^\infty \log_e |S(j\omega)| d\omega = \int_0^\infty \log_e \frac{1}{|1 + L(j\omega)|} d\omega = \pi \sum \text{Re } p_k$$
Announcements

Homework #8 is due on Friday, 2 pm
- In class or HW slot (102 STL)

Office hours this week
- Wed, 3-4 pm, 243 ANB
- Thu, 7-9 pm, 106 ANB

Final exam
- Out on 4 Dec (Fri)
- Due on 11 Dec by 5 pm: turn in to Nikki (109 Steele) or HW slot (102 STL)
- Final exam review: 4 Dec from 2-3 pm, 105 Annenberg
- Office hours during study period
  - 7 Dec (Mon), 3-5 pm
  - 8 Dec (Tue), 3-5 pm
- Piazza will be “read only” starting at ~8 pm on 8 Dec